Heterogeneity and Shocks

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Heterogeneity so far

- **Random Search:**
  - Wage posting
    - Firm heterogeneity: no wage distribution, Diamond Paradox
    - Worker heterogeneity: Albrecht & Axell (1984), partial equilibrium model, get wage distribution
- **Competitive Search:**
  - Firm heterogeneity: Moen (1997), get wage distribution

Today

- Look at random search with bargaining
  - Ex ante firm heterogeneity and ex post match productivity
  - When is the model tractable with shocks?
- Two equilibrium solutions for stochastic search models
  - Rank preserving equilibrium: random search models with shocks
  - Block recursive equilibrium: directed search models with shocks

Random Search: ex ante firm heterogeneity

- **Environment**
  - Random search, i.e. one matching function no information prior to search
  - Standard DMP setup with wage bargaining
  - Exists a distribution of firm productivities $y \sim F(y)$
- Problem: Free entry can not hold for each productivity simultaneously

Random Search: ex ante firm heterogeneity

- **Workers value functions**

$$rU = b + p(\theta)\left[\int_y E(y) - U \; dF(y)\right]$$

$$rE(y) = w(y) - \delta[U - E(y)]$$

- **Firm value functions**

$$rV(y) = -\kappa + q(\theta)[J(y) - V(y)]$$

$$rJ(y) = y - w(y) + \delta[V(y) - J(y)]$$

- **Bargaining Solution**

$$w(y) = \gamma y + (1-\gamma) rU$$

Random Search: ex ante firm heterogeneity

- **Free entry:** $V(y) = 0 \;\; \forall y$

$$\frac{\kappa}{q(\theta)} = \frac{y - w(y)}{r + \delta} \;\; \forall y$$

- $y - w(y)$ must be constant w.r.t. y, but

$$y - w(y) = (1 - \gamma)(y - rU)$$

- Free entry can not hold for each productivity simultaneously, need another equilibrium condition.
  - ex. like in Moen (1997) assume firms pay entry fee χ, then observe productivity $\Rightarrow E_y[V(y)] = \chi$

Random Search: ex post match heterogeneity

- An alternative is to assume ex post match heterogeneity
- Firms productivity y is fixed
- After matching firm and worker pull a match specific productivity $z \sim F(z)$, if match ends productivity is lost.
- Free entry can now solve the equilibrium

Random Search: ex post match heterogeneity

- **Workers value functions**

$$rU = b + p(\theta)\left[\int_{z_R} E(z) - U \; dF(z)\right]$$

$$rE(z) = w(z) - \delta[U - E(z)]$$

- **Firm value functions**

$$rV = -\kappa + q(\theta)\left[\int_{z_R} J(z) - V \; dF(z)\right]$$

$$rJ(z) = (y + z) - w(z) + \delta[V - J(z)]$$

- **Bargaining Solution**

$$w(z) = \gamma(y + z) + (1 - \gamma)rU$$

Random Search: ex post match heterogeneity

- **Reservation match productivity:** workers accept job as long as $w(z) \geq rU$, with $w(z_R) = rU$

$$w(z_R) = \gamma(y + z_R) + (1 - \gamma)rU \Rightarrow z_R = rU - y$$

- **Free entry:** rhs is now constant

$$\frac{\kappa}{q(\theta)} = \int_{z_R} \frac{(y + x) - w(z)}{r + \delta} \, dF(z)$$

- Note: the free entry condition now depends on the distribution of productivities. At this point it is tractable b/c F is exogenous.

When does this become a difficult problem?

- Consider a model with on-the-job search (OJS) (Burdett-Mortensen)
  - Here we had an equilibrium wage *offer* distribution $F(w)$ and an equilibrium wage distribution $G(w)$
  - $G(w)$ was the probability a worker was employed at wage $\leq w$
- The evolution of $G(w)$, with endogenous contact rate $p(\theta)$

$$\frac{\partial G(w,t)}{\partial t} = p(\theta)[F(w) - F(R)]u - [\delta + p(\theta)(1 - F(w))]G(w)(1-u)$$

all of these things depend on θ

When does this become a difficult problem?

- **Free entry condition**

$$\frac{\kappa}{q(\theta)} = \text{ expected profits}$$
$$= \text{[acceptance probability]} \times \text{[value added from match]}$$

- **Without OJS:** the acceptance probability was 1 in equilibrium
  - firms only bump into unemployed workers
  - unemployed workers accept wage above reservation wage
  - no firms offers below reservation wage in equilibrium

When does this become a difficult problem?

- **Free entry condition**

$$\frac{\kappa}{q(\theta)} = \text{ expected profits}$$
$$= [\text{acceptance probability}] \times [\text{value added from match}]$$

- **With OJS:** the acceptance probability depends on who they bump into
  - unemployed workers always accept
  - employed workers only accept if offer is better than current offer
  - who they bump into depends on u and $G(w)$, which both depend on θ

When does this become a difficult problem?

- **With OJS in steady state**, i.e. no shocks
  - $\partial G(w,t)/\partial t = 0$, still somewhat tractable
  - have an equation for $G(w)$ in steady state
  - have an equation for u in steady state
- **With OJS with shocks**
  - $\partial G(w,t)/\partial t \neq 0$ and depends on the evolution of $\theta(t)$, $u(t)$, and $R(t)$
  - to solve free entry we need entire evolution of $G(w,t)$ and $u(t)$

Moscarini & Postel-Vinay (2013)

- Solve a stochastic OJS model a la Burdett-Mortensen.
  - today with exogenous contact rate
  - see paper for endogenous contact rate
  - prove the existence, uniqueness, and efficiency of a Rank Preserving Equilibrium (RPE)
  - RPE is the key that makes these problems manageable

Moscarini & Postel-Vinay (2013)

- **Environment** of exogenous contact rate model
  - time is discrete, everyone discounts at β
  - there exists an underlying stochastic process, ω_t which evolves according to a first-order Markov process
  - firms heterogeneity in productivity $p \sim \Gamma(p)$, final output is $\omega_t p$
  - exogenous separations $\delta_t = \delta(\omega_t)$
  - exogenous job finding prob $\lambda_t = \lambda(\omega_t)$ while unemployed
  - exogenous job finding prob $s\lambda_t$ while employed
  - unemployed receive $b_t = b(\omega_t)$

Moscarini & Postel-Vinay (2013)

- **Timing**

  1. new state is realized ω_t
  2. employed can quit to unemployment
  3. jobs are destroyed exogenously δ_t
  4. remaining employed receive outside offer with probability $s\lambda_t$ and decided to accept or reject
  5. previously unemployed workers receive job offer with probability λ_t and decide to accept or reject
  6. production takes place and payments are made, wage and b_t

Moscarini & Postel-Vinay (2013)

- **Firms Strategies**
  - Firms choose and commit to employment contacts, i.e. a schedule of state contingent wages
  - Maximizes discounted profits s.t. other firms contracts
  - All workers in a firm get the same wage
- The employment contract
  - $V_t(p)$ the value a worker gets at time t working for a firm with productivity p
  - the wage function that implements V maximizes $t = 0$ discounted firm profits

Moscarini & Postel-Vinay (2013)

- **Equilibrium Objects** (t denotes current value given aggregate state)
  - a value of unemployment in each period U_t
  - an employment value offer distribution $F_t(W)$
  - a distribution of earned employment values $G_t(W)$
  - unemployment rate u_t

Moscarini & Postel-Vinay (2013)

- **Worker value functions**
  - Unemployment

$$U_t = b_t + \beta E_t \left[(1-\lambda_t) U_{t+1} + \lambda_t \int max\{x, U_{t+1}\} \, dF_{t+1}(x) \right]$$

  - Employment

$$W_t = w_t + \beta E_t \Big[\delta_{t+1} U_{t+1} + (1-\delta_{t+1})(1-s\lambda_{t+1}) W_{t+1}$$
$$+ (1-\delta_{t+1}) s\lambda_{t+1} \int_{W_{t+1}} x - W_{t+1} \; dF(x) \Big]$$

Moscarini & Postel-Vinay (2013)

- Labor supply to firm of type p

$$
\begin{aligned}
L_{t+1}(p) = {} & L_t(p)(1-\delta_{t+1})[1 - s\lambda_{t+1}[1 - F_{t+1}(V_{t+1}(p))]] \\
& + \lambda_{t+1}[1 - N_t(\bar{p})] \\
& + s\lambda_{t+1}(1-\delta_{t+1})N_t(\bar{p})G_{t+1}(V_{t+1}(p))
\end{aligned}
$$

- Total employment at firms less or equal to p

$$
N_t(p) = \int_{\underline{p}}^{p} L_t(p)\, d\Gamma(p)
$$

- Unemployment

$$
u_t = 1 - N_t(\bar{p})
$$

Moscarini & Postel-Vinay (2013)

- Firms problem: to maximize expected discounted profits Π_0
- Let $\bar{V}$ be the value the firm promised in period $t-1$ to deliver in period t, then we can write the problem recursively s.t. offering at least $\bar{V}$

$$\Pi(\bar{V}) = \max_{w_t, W_{t+1} \geq U_{t+1}} (\omega_t p - w_t) L_t + \beta E_t[\Pi_{t+1}(W_{t+1})]$$

$$\text{s.t. } \bar{V} = w_t + \beta E_t \Big[\delta_{t+1} U_{t+1} + (1-\delta_{t+1})(1 - s\lambda_{t+1}) W_{t+1} + (1-\delta_{t+1}) s\lambda_{t+1} \int_{W_{t+1}} x - W_{t+1} \; dF(x) \Big]$$

Moscarini & Postel-Vinay (2013)

- This problem can be rewritten (see paper) to show that the solution does not depend on the current promised value $\bar{V}$
- Intuition
  - at time t firm offers state contingent W_{t+1} to maximize profits Π_{t+t}
  - then to deliver the W_t it promised last period it adjusts w_t
  - because it is offering W_t in period t which was chosen optimally in period $t-1$, profits in period t are maximized.

Moscarini & Postel-Vinay (2013)

- To solve all this we still need the offer distribution $F_t(W)$ and the earned value distribution $G_t(W)$, both of these show up in L_t and U_t

$$F_t(W) = \int_{\underline{p}}^{\bar{p}} \mathbb{I}\{V_t(p) \leq W\}\ d\Gamma(p)$$

$$G_t(W) = \frac{1}{N_t(p)} \int_{\underline{p}}^{\bar{p}} \mathbb{I}\{V_t(p) \leq W\}\ dN_t(p)$$

- This is hard to solve, F and G depend on V each period, but to solve for V from firm's problem we need to know F and G

Moscarini & Postel-Vinay (2013)

- **Rank Perserving Equilibrium**: a Markov equilibrium V where, on the equilibrium path, a more productive firm always offers its workers a higher continuation value $V_{t+1}(p) = V(p, L_t(p), \omega_{t+1}, N_t)$ is increasing in p, including the effect of p on current firm size $L_t(p)$.

- In a RPE we have

$$F_t(V_t(p)) \equiv \Gamma(p)$$

$$G_t(V_t(p)) = \frac{N_{t-1}(p)}{N_{t-1}(\bar{p})}$$

Moscarini & Postel-Vinay (2013)

- **RPE Properties**
  - labor allocations are constrained efficient, i.e. all movements from U to E are efficient, all E to E movements are up the job ladder
  - Uniqueness: there exists at most one RPE
  - Existence: and more productive firms are initially weakly larger ($L_0(p)$ is non-decreasing)
- See paper for a condition on the optimal contract.
- See paper for endogenous contact rates

Shocks in a directed search model

- Now let's look at the same type of model in a directed search framework
  - heterogeneity in production
  - shocks to aggregate productivity
  - on the job search
- The equilibrium will be block recursive
  - block 1: decisions rules and tightness can be solved without knowing the distribution of workers across unemployment and employment productivities
  - block 2: the distribution of workers is solved for using the decision rules

Menzio & Shi (2011)

- **Environment**
  - Time is discrete
  - Everyone discounts at β
  - Workers have a period utility function $\nu(\cdot)$, weakly concave
  - Aggregate productivity is $y \in \{y_1, ..., y_{N_y}\}$
    - drawn from $\Phi(\hat{y}|y)$
  - Idiosyncratic match productivity $z \in \{z_1, ..., z_{N_z}\}$
    - drawn from $\Phi(\hat{z}|z)$
  - Final production $y + z$

Menzio & Shi (2011)

- **Environment cont.**
  - There exist submarkets which are indexed by the lifetime utility x that the worker receives
  - Each submarket has a matching technology as a function of tightness θ
    - job finding probability $p(\theta)$
    - job filling probability $q(\theta)$
  - δ is the separation probability
- **Aggregate state:** $\psi(y, u, g) \in \Psi$
  - y draw of the aggregate productivity
  - $u \in [0, 1]$the measure of unemployed workers
  - $g(V, z)$ measure of workers employed at jobs that gives them lifetime utility $\leq V$ and have an idiosyncratic component of productivity $\leq z$

Menzio & Shi (2011)

- **Workers**
  - λ_u probability they can search while unemployed
  - λ_e probability they can search while employed
  - get b while unemployed
- **Frims**
  - post vacancies in a submarket at cost k
  - choose an employment contract that give the worker his promised utility and maximizes their discounted profits
    - dynamic wage
    - fixed wage contract
  - offers work a two point lottery over the employment contract that is drawn at the begining of the match

Menzio & Shi (2011)

- **Timing**
  1. a new y is drawn and a new z is drawn for all employed
  2. Separation
     - exogenous separation
     - employed can choose to quit
  3. Search
     - previously unemployed workers w/ prob λ_u
     - still employed w/ prob λ_e
     - newly unemployed do not
  4. Matching
  5. Production and payments

Menzio & Shi (2011)

- **Employed worker**: employed at a job with value V
  - search value function

$$R(V, \Psi) = \max_{x \in X} \; p(\theta(x, \Psi))(x - V)$$

  - decision rule

$$m(V, \Psi)$$

- **Unemployed worker**
  - value function

$$U(\Psi) = b + \beta E_{\hat{\Psi}}[U(\hat{\Psi}) + \lambda_u \max\{0, R(U(\hat{\Psi}), \hat{\Psi})\}]$$

  - decision rule

$$m(U, \Psi)$$

Menzio & Shi (2011)

- **Fixed wage employment contract:** firms commit to a constant wage throughout employment
  - offer the worker a two point lottery over employment contract
  - wage can depend on the outcome of lottery but fixed after
  - lottery maximizes firm's discounted profits while guaranteeing the worker the value posted in the submarket

Menzio & Shi (2011): Fixed wage contract

- $H(w, \Psi)$: workers discounted lifetime utility at wage w and state of the world Ψ

$$H(w, \Psi) = w + \beta E_{\hat{\Psi}}\{d(\hat{\Psi})U(\hat{\Psi}) - (1 - d(\hat{\Psi}))[H(w, \hat{\Psi}) \\ + \lambda_e \max\{0, R(H(w, \hat{\Psi}), \hat{\Psi})\}]\}$$

$$d(\hat{\Psi}) = \begin{cases} \delta & U(\hat{\Psi}) \leq H(w, \hat{\Psi}) + \lambda_e \max\{0, R(H(w, \hat{\Psi}), \hat{\Psi})\} \\ 1 & \text{otherwise} \end{cases}$$

- Let $h(V, \Psi)$ be the solution to the wage, w, such that $H(w, \Psi) = V$

Menzio & Shi (2011): Fixed wage contract

- $K(w, \Psi, z)$: firms lifetime discounted profits of hiring a worker at wage w in the state of the world Ψ and match specific draw z

$$\begin{aligned} K(w,\Psi, z) &= y + z - w \\ &\quad + \beta E_{\hat{\Psi},\hat{z}}\{(1 - d(\hat{\Psi}))[1 - \lambda_e \tilde{p}(H(w, \hat{\Psi}), \hat{\Psi})]K(w, \hat{\Psi}, \hat{z})\} \end{aligned}$$

- $d(\hat{\Psi})$ as before

- $\tilde{p}(\cdot)$ is job finding prob. in the optimal submarket

Menzio & Shi (2011): Fixed wage contract

- $J(V, \Psi, z_0)$: firms lifetime discounted profits matching in submarket $x = V$ in the state of the world Ψ and match specific draw z_0

$$J(V, \Psi, z_0) = \max_{\pi_i, \tilde{V}_i} \sum_{i=1}^{2} \pi_i K(h(\tilde{V}_i, \Psi), \Psi, z_0)$$

$$\text{s.t. } \pi_i \in [0, 1],\ \ \tilde{V}_i \in X, \quad \text{for } i = 1, 2$$

$$\pi_1 + \pi_2 = 1,\ \ \pi_1 \tilde{V}_1 + \pi_2 \tilde{V}_2 = V$$

- Let $c(V, \Psi, z_0)$ be the optimal policy function

Menzio & Shi (2011)

- **Free entry:** firms post vacancies in submarkets until expected profit equals expected cost

$$k = q(\theta(x, \Psi))J(x, \Psi, z_0) \ \forall x$$

Menzio & Shi (2011)

- **Recursive Equilibrium:** a market tightness function $\theta : X \times \Psi \to \mathbb{R}^+$, a search value function $R : X \times \Psi \to \mathbb{R}$, a search policy function: $m : X \times \Psi \to X$, an unemployment value function $U : \Psi \to X$, a firm's value function $J : X \times \Psi \times z \to \mathbb{R}$, a contract policy function $c : X \times \Psi \times Z \to C$ and a transition probability function for the aggregate state of the economy $\Phi_{\hat{\Psi}} : \Psi \times \Psi \to [0, 1]$. These functions satisfy the following requirements:
    - θ satisfies free entry condition
    - R maximizes worker's search problem, with optimal policy m
    - U satisfies unemployed workers problem
    - J maximizes firm profits, with optimal policy c
    - $\Phi_{\hat{\Psi}}$ is derived from c, and m

Menzio & Shi (2011)

- **Block Recursive Equilibrium:** a recursive equilibrium such that the functions $\{\theta, R, m, U, J, c\}$ depend on the aggregate state of the economy, Ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across employment states,(u, g).
    - for each y can solve for $\{\theta, R, m, U, J, c\}$
    - the using m, c and Φ_y, Φ_z you can solve for the transition probabilities, Φ_Ψ, of the aggregate state $\Psi = \{y, u, g\}$

Existence and Properties

- Menzio & Shi (2011): the existence of a BRE does not depend on the type of contract, fixed vs dynamic
  - does not depend on completeness of contracts
- Shi (2009): the existence of a BRE does not depend on risk neutrality of workers
- Menzio & Shi (2010): prove existence of BRE for ex ante worker heterogeneity
- Menzio & Shi (2014): efficiency and uniqueness