Guess and Verify with Log Utility: Step-by-Step Derivation of Constants A and B

Problem Setup

Consider the infinite horizon consumption-savings problem with constant income $y > 0$:

$$V(a) = \max_{c \geq 0} \{\ln(c) + \beta V((1+r)a + y - c)\} \tag{1}$$

subject to the budget constraint:

$$a' = (1+r)a + y - c \tag{2}$$

where:

- a: current assets
- c: consumption choice
- a': next period assets
- $\beta \in (0,1)$: discount factor
- $r > 0$: interest rate
- $y > 0$: constant income per period

The Guess

Based on the log utility function, we guess that the value function has the form:

$$V(a) = A \ln(a + B) \tag{3}$$

where $A > 0$ and $B > 0$ are constants to be determined.

Why This Form?

- The period utility is $\ln(c)$, so a log value function is natural
- The term $(a + B)$ allows us to handle the income stream y
- We expect B to be related to the present value of the income stream
- We expect A to be related to the discount factor β

Step 1: Substitute the Guess into the Bellman Equation

Substituting our guess $V(a) = A\ln(a+B)$ into the Bellman equation:

$$A\ln(a+B) = \max_{c\geq 0}\{\ln(c) + \beta A\ln((1+r)a + y - c + B)\} \tag{4}$$

Step 2: Take the First-Order Condition

The first-order condition with respect to c is:

$$\frac{\partial}{\partial c}[\ln(c) + \beta A\ln((1+r)a + y - c + B)] = 0 \tag{5}$$

Computing the derivatives:

$$\frac{1}{c} + \beta A \cdot \frac{1}{(1+r)a + y - c + B} \cdot (-1) = 0 \tag{6}$$

Simplifying:

$$\frac{1}{c} = \frac{\beta A}{(1+r)a + y - c + B} \tag{7}$$

Step 3: Solve for Optimal Consumption

From the first-order condition:

$$\frac{1}{c} = \frac{\beta A}{(1+r)a + y - c + B} \tag{8}$$

Cross-multiplying:

$$(1+r)a + y - c + B = \beta A c \tag{9}$$

Rearranging to solve for c:

$$(1+r)a + y + B = c + \beta A c = c(1+\beta A) \tag{10}$$

Therefore, the optimal consumption is:

$$c^*(a) = \frac{(1+r)a + y + B}{1+\beta A} \tag{11}$$

Step 4: Substitute Optimal Consumption Back into Bellman Equation

Now we substitute $c^*(a)$ back into the right-hand side of the Bellman equation:

$$A\ln(a+B) \tag{12}$$
$$= \ln\left(\frac{(1+r)a + y + B}{1+\beta A}\right) + \beta A\ln\left((1+r)a + y - \frac{(1+r)a + y + B}{1+\beta A} + B\right) \tag{13}$$

Simplify the Second Logarithm. Let's compute the argument of the second logarithm:

$$
\begin{aligned}
&(1+r)a + y - \frac{(1+r)a+y+B}{1+\beta A} + B && (14)\\
&= (1+r)a + y + B - \frac{(1+r)a+y+B}{1+\beta A} && (15)\\
&= ((1+r)a+y+B)\left(1 - \frac{1}{1+\beta A}\right) && (16)\\
&= ((1+r)a+y+B)\left(\frac{1+\beta A - 1}{1+\beta A}\right) && (17)\\
&= ((1+r)a+y+B)\left(\frac{\beta A}{1+\beta A}\right) && (18)
\end{aligned}
$$

The Bellman equation becomes:

$$
\begin{aligned}
A\ln(a+B) = \ln&\left(\frac{(1+r)a+y+B}{1+\beta A}\right) && (19)\\
&+ \beta A \ln\left(((1+r)a+y+B)\frac{\beta A}{1+\beta A}\right) && (20)
\end{aligned}
$$

Using the logarithm property $\ln(xy) = \ln(x) + \ln(y)$:

$$
\begin{aligned}
A\ln(a+B) = \ln&\left(\frac{(1+r)a+y+B}{1+\beta A}\right) && (21)\\
&+ \beta A \ln((1+r)a+y+B) + \beta A \ln\left(\frac{\beta A}{1+\beta A}\right) && (22)
\end{aligned}
$$

Using $\ln(x/y) = \ln(x) - \ln(y)$:

$$
\begin{aligned}
A\ln(a+B) = \ln&((1+r)a+y+B) - \ln(1+\beta A) && (23)\\
&+ \beta A \ln((1+r)a+y+B) + \beta A \ln\left(\frac{\beta A}{1+\beta A}\right) && (24)
\end{aligned}
$$

Collecting terms with $\ln((1+r)a+y+B)$:

$$
\begin{aligned}
A\ln(a+B) = (1+\beta A)&\ln((1+r)a+y+B) - \ln(1+\beta A) && (25)\\
&+ \beta A \ln\left(\frac{\beta A}{1+\beta A}\right) && (26)
\end{aligned}
$$

Step 5: Match Coefficients

For this equation to hold for all values of a, we need the coefficients of $\ln(\cdot)$ terms and the constant terms to match on both sides. We have the identity:

$$A\ln(a+B) = (1+\beta A)\ln((1+r)a+y+B) + \text{constant terms} \tag{27}$$

For this equation to hold we need that

$$A = (1+\beta A) \tag{28}$$
$$(1-\beta)A = 1 \tag{29}$$
$$A = \frac{1}{1-\beta} \tag{30}$$

Since $A = \frac{1}{1-\beta}$, we have $1+\beta A = 1+\frac{\beta}{1-\beta} = \frac{1-\beta+\beta}{1-\beta} = \frac{1}{1-\beta} = A$.
So our equation becomes:

$$A\ln(a+B) = A\ln((1+r)a+y+B) + \text{constant terms} \tag{31}$$

For this to hold for all a, we need:

$$\ln(a+B) = \ln((1+r)a+y+B) + \frac{\text{constant terms}}{A} \tag{32}$$

Taking exponentials:

$$a+B = e^{\frac{\text{constant terms}}{A}} \cdot ((1+r)a+y+B) \tag{33}$$

For this to be an identity in a, we need the coefficient of a to be 1 on both sides:

$$1 = e^{\frac{\text{constant terms}}{A}} \cdot (1+r) \tag{34}$$
$$e^{\frac{\text{constant terms}}{A}} = \frac{1}{1+r} \tag{35}$$

And for the constant terms:

$$B = \frac{1}{1+r} \cdot (y+B) \tag{36}$$

Solving for B:

$$B = \frac{y+B}{1+r} \tag{37}$$
$$B(1+r) = y+B \tag{38}$$
$$B(1+r) - B = y \tag{39}$$
$$Br = y \tag{40}$$
$$B = \frac{y}{r} \tag{41}$$

Final Answer

We have determined both constants:

$$\boxed{A = \frac{1}{1-\beta}} \quad \text{and} \quad \boxed{B = \frac{y}{r}} \tag{42}$$

Economic Interpretation

- $A = \frac{1}{1-\beta}$: This represents the effect of discounting. As $\beta \to 1$ (more patient), $A \to \infty$ (higher value of future utility).
- $B = \frac{y}{r}$: This is the present value of the infinite income stream, also called "human wealth."

Final Value and Policy Functions

With $A = \frac{1}{1-\beta}$ and $B = \frac{y}{r}$, we get:

Value Function

$$\boxed{V(a) = \frac{1}{1-\beta} \ln\left(a + \frac{y}{r}\right)} \tag{43}$$

Policy Function

$$\begin{aligned}
c^*(a) &= \frac{(1+r)a + y + B}{1 + \beta A} && (44) \\
&= \frac{(1+r)a + y + \frac{y}{r}}{1 + \beta \cdot \frac{1}{1-\beta}} && (45) \\
&= \frac{(1+r)a + y(1 + \frac{1}{r})}{\frac{1-\beta+\beta}{1-\beta}} && (46) \\
&= \frac{(1+r)a + \frac{y(r+1)}{r}}{\frac{1}{1-\beta}} && (47) \\
&= (1-\beta)(1+r)a + (1-\beta)\frac{y(1+r)}{r} && (48) \\
&= (1-\beta)(1+r)\left(a + \frac{y}{r}\right) && (49)
\end{aligned}$$

Therefore:

$$\boxed{c^*(a) = (1-\beta)(1+r)\left(a + \frac{y}{r}\right)} \tag{50}$$

Economic Interpretation of Policy Function

The optimal consumption is a constant fraction $(1-\beta)(1+r)$ of total wealth, where total wealth is the sum of:

- Financial wealth: a
- Human wealth: $\frac{y}{r}$ (present value of income stream)

Euler Equation Check

The Euler equation for this problem is:

$$\frac{1}{c_t} = \beta(1+r)\frac{1}{c_{t+1}} \tag{51}$$

From our policy function:

$$c_t = (1-\beta)(1+r)\left(a_t + \frac{y}{r}\right) \tag{52}$$

$$c_{t+1} = (1-\beta)(1+r)\left(a_{t+1} + \frac{y}{r}\right) \tag{53}$$

With $a_{t+1} = (1+r)a_t + y - c_t$, we can verify that the Euler equation holds (verification omitted for brevity but can be checked algebraically).