Neoclassical Growth with Exogenous Labor

Ramsey-Cass-Koopmans Model

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Overview

Last lecture: Partial equilibrium analysis

- Consumption-savings with exogenous income and interest rate
- Agent takes y and r as given
- Optimal consumption: $c^*(a) = (1-\beta)(1+r)(a+\frac{y}{r})$

Today: General equilibrium analysis

- Endogenous income and returns through capital accumulation
- Agent chooses consumption *and* investment
- Output produced using accumulated capital
- Interest rate determined by marginal product of capital

Key insight: Saving today increases tomorrow's productive capacity, affecting both future income and returns to saving.

Ramsey-Cass-Koopmans Model

Key features:

- **Infinite-horizon** representative agent
- **Endogenous** capital accumulation
- **Neoclassical** production function
- **Perfect competition** and market clearing
- **No uncertainty** (deterministic environment)

Central questions:

- How much should society save vs. consume?
- What determines long-run capital stock and consumption?
- How does the economy transition to long-run equilibrium?
- What factors affect growth and accumulation?

Applications: Optimal growth, fiscal policy, development economics

Historical Context

Frank Ramsey (1928): "A Mathematical Theory of Saving"

- Normative question: How much should a nation save?
- Mathematical framework for optimal saving
- Foundation of modern growth theory

David Cass (1965) & Tjalling Koopmans (1965):

- Rigorous infinite-horizon dynamic programming approach
- Competitive equilibrium interpretation
- Decentralization results

Modern relevance:

- Benchmark model in macroeconomics
- Foundation for DSGE models
- Policy analysis framework
- Understanding long-run growth patterns

The Economic Environment

Representative agent lives forever and chooses consumption and investment.

Preferences:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- c_t: consumption at time t
- $\beta \in (0, 1)$: discount factor
- $u(\cdot)$: period utility function (increasing, concave)

Technology:

$$y_t = f(k_t)$$

- y_t: output at time t
- k_t: capital stock at time t
- $f(\cdot)$: production function (increasing, concave)

Note: Labor is exogenous and normalized to 1, so $f(k_t)$ represents output per worker.

Technology Assumptions

Neoclassical production function $y = f(k)$ satisfies:

1. **Positive and diminishing marginal product:**

$$f'(k) > 0, \quad f''(k) < 0$$

2. **Inada conditions:**

$$\lim_{k \to 0} f'(k) = \infty, \quad \lim_{k \to \infty} f'(k) = 0$$

3. **Standard normalization:**

$$f(0) = 0$$

Economic interpretation:

- More capital increases output but at diminishing rate
- Very low capital has very high marginal productivity
- Very high capital has very low marginal productivity
- Ensures interior solutions and convergence

Capital Accumulation

Law of motion for capital:

$$k_{t+1} = (1-\delta)k_t + i_t$$

where:

- i_t: gross investment at time t
- $\delta \in (0,1)$: depreciation rate
- $(1-\delta)k_t$: undepreciated capital from period t

Resource constraint:

$$c_t + i_t = f(k_t)$$

Output can be either consumed or invested (no waste).

Combining these:

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$$

Economic interpretation: Today's consumption choice determines tomorrow's capital stock, which affects future productive capacity.

Representative Household Problem

$$\max_{\{c_t,k_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to:

$$c_t + k_{t+1} = r_t k_t + (1-\delta)k_t \quad \forall t \tag{1}$$

$$k_0 \text{ given} \tag{2}$$

$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0 \quad \text{(transversality)} \tag{3}$$

Budget Constraint:

- agents own the capital and get a return each period
- agent income $(r_t k_t)$ must equal consumption and investment $(c_t + i_t)$
- replacing i_t with the law of motion of capital gives (1)

Representative Firm Problem

Each period the firm maximizes profits taking price (r_t) as given

$$\pi_t = \max_{k_t^d} f(k_t^d) - r_t k_t^d$$

where k_t^d is capital demand.

Equilibrium vs. Optimum

Optimum: The best possible outcome according to some criterion (usually maximizing welfare or utility)

- Perspective: Normative - what should happen
- Determined by: A social planner who can control all variables
- Criterion: Usually maximizes total social welfare
- Single optimization problem

Equilibrium: A state where no agent has an incentive to unilaterally change their behavior

- Perspective: Positive - what will happen given how agents actually behave
- Determined by: Individual optimization by all agents simultaneously
- Criterion: Each agent maximizes their own objective, taking others' actions as given
- Multiple optimization problems (households, firms)

First Welfare Theorem: Under perfect competition and standard assumptions, competitive equilibrium equals social optimum.

Defining Economic Equilibrium

General definition: An equilibrium is a state where all economic agents optimize given their constraints, and all markets clear.

Key components:

1. **Individual optimization:** Each agent chooses actions to maximize their objective
2. **Market clearing:** Supply equals demand in all markets
3. **Consistency:** Agents' beliefs about prices and others' actions are correct
4. **No incentive to deviate:** Given equilibrium prices and others' actions, no agent wants to change their behavior

In the Ramsey model context:

- Households maximize utility subject to budget constraints
- Firms maximize profits subject to technology constraints
- Capital and goods markets clear
- Price expectations are fulfilled

Equilibrium Concepts

Sequential Market Equilibrium: An equilibrium where there's a complete set of markets for goods at every date (and state), all trading occurs at time 0, and prices clear all markets simultaneously.

- outlines a set of prices and allocations that result from agent optimization and market clearing
- think Lagrangian Method from first lecture

Recursive Equilibrium: An equilibrium characterized by time-invariant policy functions and value functions, where current decisions depend only on current state variables (not the entire history).

- outlines functions (value, policy, pricing) that result from agent optimization and market clearing
- think dynamic program from first lecture

Sequential Markets: Basic Setup

Market structure: Markets open sequentially over time, one period at a time.

At each date t:

- Capital rental market opens with price r_t
- Goods market opens with price $p_t = 1$ (numeraire)
- Agents trade based on current information
- Markets close, time moves to $t + 1$

Key features:

- **Spot markets only:** No trading of future delivery contracts
- **Sequential decision-making:** Agents decide period by period
- **Price-taking behavior:** Agents take current prices as given
- **Perfect foresight:** Agents correctly anticipate future prices

Sequential Markets

Household's first-order condition:

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + r_{t+1}]$$

Firm's first-order condition:

$$r_t = f'(k_t)$$

Combine to get the **Euler equation**

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})]$$

which tells us the rate at which consumption changes given $f'(k_{t+1})$. So we also need to know how capital changes and an initial condition k_0. Combining the law of motion of capital and the resource constraints gives us

$$k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t$$

Sequential Markets: Equilibrium Definition

A Sequential Markets Equilibrium is a sequence of:

- Allocations: $\{c_t, k_{t+1}, k_t^d\}_{t=0}^{\infty}$
- Prices: $\{r_t\}_{t=0}^{\infty}$

such that:

1. **Household optimization:** $\{c_t, k_{t+1}\}$ solves household's problem
2. **Firm optimization:** k_t^d solves firm's problem each period
3. **Capital market clearing:** $k_t^d = k_t$ for all t
4. **Goods market clearing:** $c_t + i_t = f(k_t)$ for all t

Challenge: Agents must have perfect foresight about future prices $\{r_s\}_{s=t+1}^{\infty}$ to solve their optimization problems at date t.

Motivation for Recursive Approach

Challenges with sequential markets:

- Agents need to predict infinite sequence of future prices
- Computationally complex: infinite-dimensional price space
- Difficult to analyze stability and uniqueness
- Hard to compute numerically

Recursive approach solution:

- Express everything in terms of current state variables
- Prices depend only on current aggregate state
- Reduces infinite-dimensional problem to finite-dimensional
- Enables dynamic programming techniques

Key idea: Prices can be expressed as functions of current state rather than time.
State variable in Ramsey model: Current aggregate capital stock K_t.

Recursive Equilibrium: Setup

Aggregate state: K (per-capita capital since population normalized to 1)
Price functions:

- Rental rate: $r(K)$

Aggregate law of motion:

$$K' = G(K)$$

where $G(\cdot)$ is the aggregate policy function to be determined in equilibrium.

Individual state: Current capital holdings k
Individual problem: Given aggregate state K and law of motion $K' = G(K)$:

$$V(k, K) = \max_{c,k'} \left\{ u(c) + \beta V(k', G(K)) \right\}$$

subject to: $c + k' = r(K)k + (1 - \delta)k$

Recursive Equilibrium: Individual Optimization

Individual policy functions: $c = g_c(k, K)$ and $k' = g_k(k, K)$
First-order conditions:

$$u'(g_c(k, K)) = \beta V_k(g_k(k, K), G(K))$$

Envelope condition:

$$V_k(k, K) = u'(g_c(k, K))[r(K) + 1 - \delta]$$

Combining:

$$u'(g_c(k, K)) = \beta[r(G(K)) + 1 - \delta]u'(g_c(g_k(k, K), G(K)))$$

$$u'(c) = \beta[r(G(K)) + 1 - \delta]u'(c')$$

Note: Individual takes aggregate law of motion $G(K)$ as given.

Recursive Equilibrium: Consistency Conditions

Representative agent assumption: Individual capital equals aggregate capital supplied: $k = K$.

Market clearing in capital market:

$$r(K) = f'(K) \Rightarrow K = K^d$$

Market clearing in goods market:

$$g_c(K, K) + g_k(K, K) = f(K) + (1 - \delta)K$$

Consistency of aggregate law of motion:

$$G(K) = g_k(K, K)$$

Key insight: Aggregate behavior must be consistent with individual optimization.

Recursive Competitive Equilibrium: Definition

A Recursive Competitive Equilibrium consists of:

- Value function: $V(k, K)$
- Policy functions: $g_c(k, K)$, $g_k(k, K)$
- Price functions: $r(K)$
- Aggregate law of motion: $G(K)$

such that:

1. **Individual optimization:** Given $(r(K), w(K), G(K))$, the value and policy functions solve the individual's dynamic programming problem
2. **Market clearing:** $r(K) = f'(K)$
3. **Consistency:** $G(K) = g_k(K, K)$

Equivalence of the Two Approaches

Fundamental result: Under standard assumptions, sequential markets equilibrium and recursive competitive equilibrium yield identical allocations.

Why they're equivalent:

- Both implement the same first-order conditions
- Both satisfy the same market clearing conditions
- Both respect the same resource constraints
- Representative agent framework eliminates distributional issues

Mathematical equivalence:

- Sequential: $u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})]$
- Recursive: $u'(c) = \beta u'(c')[1 - \delta + f'(k')]$ where $c' = g_c(k', k')$

Both lead to: $k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t$ with same Euler equation.

Taking Stock

So far we have discussed

- given an initial capital stock k_0, **sequential markets** approach gives optimal consumption path
- **recursive approach** gives a consumption policy function which we can use to determine optimal consumption at any capital level
- given the same initial capital stock k_0, both approaches give the same answer

Now let's think about

- how capital changes over time, i.e. growing or shrinking?
- are there any cases where capital is not changing?
- how does the initial capital stock k_0 affect the dynamics?

Dynamics of the model

To understand the dynamics, we analyze the system:

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t \tag{4}$$

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})] \tag{5}$$

Steady State: A point in the system where all endogenous variables, (k, c), are constant over time.

Phase diagram: plot which shows all possible paths in (k, c) space and which ones lead to steady state.

Types of Steady States

1. **Stable steady state:**
   - Small perturbations lead back to steady state
   - Attracting in the dynamics
2. **Unstable steady state:**
   - Small perturbations lead away from steady state
   - Repelling in the dynamics
3. **Saddle point steady state:**
   - Stable in some dimensions, unstable in others
   - Unique convergent path (saddle path)

In Ramsey model: Steady state is saddle point stable with unique convergent path.

Steady State

In steady state:

$$k_{t+1} = k_t = k^* \quad \text{(constant capital)} \tag{6}$$

$$c_{t+1} = c_t = c^* \quad \text{(constant consumption)} \tag{7}$$

$$f'(k^*) = r^* \quad \text{(constant marginal product)} \tag{8}$$

Golden Rule Question

“What level of capital maximizes steady-state consumption per capita?”

Modified Golden Rule Question

“What level of capital do optimizing agents actually choose?”

The Golden Rule

Objective: Maximize steady-state consumption

In steady state $k_t = k_{t+1} = k$, so from the capital accumulation equation (6) we get

$$\max c = f(k) - \delta k$$

First-Order Condition

$$\frac{dc}{dk} = f'(k) - \delta = 0$$

Golden Rule Condition

$$f'(k_{GR}) = \delta$$

Economic intuition: Balance marginal output against marginal investment needs

The Modified Golden Rule (Ramsey Rule)

Objective: Maximize discounted lifetime utility

Note: The consumption level needed to do this comes from the Euler equation.

From the Euler equation:

$$u'(c^*) = \beta u'(c^*)[1 - \delta + f'(k^*)]$$

Since $u'(c^*) > 0$, we can divide both sides:

$$1 = \beta[1 - \delta + f'(k^*)]$$

Solving for marginal product:

$$f'(k^*) = \frac{1}{\beta} - (1 - \delta) = \frac{1 - \beta}{\beta} + \delta$$

The Modified Golden Rule (Ramsey Rule)

The steady-state condition:

$$f'(k^*) = \frac{1-\beta}{\beta} + \delta$$

can be rewritten as:

$$f'(k^*) = \rho + \delta$$

where $\rho = \frac{1-\beta}{\beta}$ is the subjective discount rate.

Economic interpretation:

- $f'(k^*)$: marginal product of capital
- $\rho + \delta$: "required return" on capital

Key relationship: Since $\rho > 0$, we have $f'(k^*) > f'(k_{GR})$, which implies $k^* < k_{GR}$ (by diminishing returns).

Economic intuition: Impatience (ρ) makes agents save less than Golden Rule

Steady-State Consumption

Once we know k^*, steady-state consumption follows from resource constraint:

$$c^* + i^* = f(k^*)$$

In steady state, investment just replaces depreciated capital:

$$k^* = (1-\delta)k^* + i^* \quad \Rightarrow \quad i^* = \delta k^*$$

Therefore:

$$c^* = f(k^*) - \delta k^*$$

Economic interpretation:

- Output: $f(k^*)$
- Replacement investment: δk^*
- Available for consumption: $f(k^*) - \delta k^*$

Key insight: Higher steady-state capital doesn't always mean higher consumption due to depreciation costs!

The Phase Diagram

A **phase diagram** is a graphical representation of the state space of a dynamic system that shows:

- The direction of movement from any given state
- Equilibrium points and their stability properties
- Trajectories showing how the system evolves over time

They system:

$$k_{t+1} = (1-\delta)k_t + f(k_t) - c_t$$
$$u'(c_t) = \beta u'(c_{t+1})[1-\delta + f'(k_{t+1})]$$

For phase diagram, define:

- $\dot{k} = k_{t+1} - k_t = f(k_t) - \delta k_t - c_t$
- $\dot{c} = c_{t+1} - c_t$ (determined by Euler equation)

The $\dot{k} = 0$ Locus

$$\dot{k} = 0 \quad \Rightarrow \quad c = f(k) - \delta k$$

At low k**:** $f'(k) > n + \delta$ (by Inada conditions)

- So $c' = f'(k) - (n + \delta) > 0$
- **Locus is upward sloping**

At high k**:** $f'(k) < n + \delta$ (diminishing returns)

- So $c' = f'(k) - (n + \delta) < 0$
- **Locus is downward sloping**

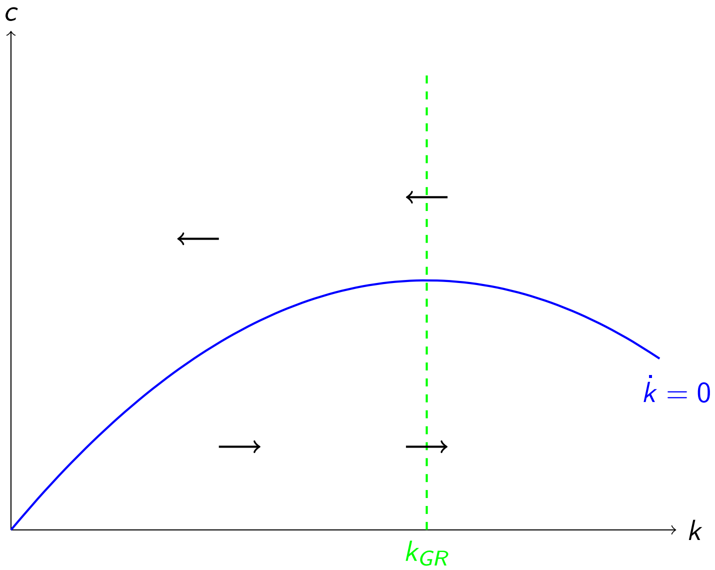
At some intermediate k^{GR}**:** $f'(k^{GR}) = n + \delta$

- So $c'(k^{GR}) = 0$
- **This is the peak** (Golden Rule capital stock!)

Dynamics

- Above locus: $c > f(k) - \delta k \Rightarrow \Delta k < 0$ (capital falls)
- Below locus: $c < f(k) - \delta k \Rightarrow \Delta k > 0$ (capital rises)

Phase Diagram: Ramsey Model Dynamics



The $\dot{c} = 0$ Locus

Consumption is constant when:

$$c_{t+1} = c_t$$

From Euler equation, this requires:

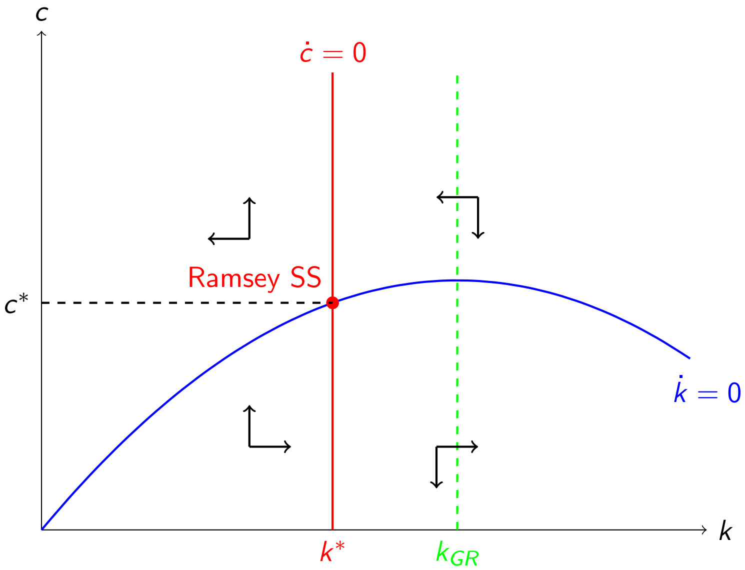
$$1 = \beta[1 - \delta + f'(k)]$$

$$f'(k) = \rho + \delta$$

Properties:

- Vertical line at k^* where $f'(k^*) = \rho + \delta$
- Independent of consumption level
- Left of line: $f'(k) > \rho + \delta \Rightarrow \Delta c > 0$ (consumption rises)
- Right of line: $f'(k) < \rho + \delta \Rightarrow \Delta c < 0$ (consumption falls)

Phase Diagram: Ramsey Model Dynamics



Phase Diagram Regions and Saddle Path

Four regions with different dynamics:

- **Region I:** $k < k^*$, high $c \rightarrow \Delta k < 0$, $\Delta c > 0$
- **Region II:** $k > k^*$, high $c \rightarrow \Delta k < 0$, $\Delta c < 0$
- **Region III:** $k < k^*$, low $c \rightarrow \Delta k > 0$, $\Delta c > 0$
- **Region IV:** $k > k^*$, low $c \rightarrow \Delta k > 0$, $\Delta c < 0$

A **saddle path** (or stable manifold) is the unique trajectory that approaches a saddle point equilibrium as time goes to infinity.

Saddle Path Stability

Key insight: Most initial conditions lead away from steady state!

Saddle path properties:

- Unique stable manifold leading to (k^*, c^*)
- Given any initial k_0, there's exactly one c_0 that leads to steady state
- All other initial consumption levels lead to explosive paths

Economic interpretation:

- **Too high initial c:** Insufficient saving $\rightarrow$ capital declines $\rightarrow$ economy collapses
- **Too low initial c:** Excessive saving $\rightarrow$ capital explodes $\rightarrow$ violates transversality
- **Saddle path c:** "Just right" balance between current and future consumption

Policy implication: Forward-looking agents must choose initial consumption optimally to avoid unstable paths.

Multiple Steady States

Can there be multiple steady states?

In the basic Ramsey model: **NO**

- Condition $f'(k^*) = \rho + \delta$ has unique solution
- Strict concavity of f ensures uniqueness
- Inada conditions guarantee interior solution

Extensions with multiple steady states:

- **Threshold effects:** Different production technologies for different capital ranges
- **External effects:** $f(k, K)$ where K is aggregate capital
- **Non-convexities:** Fixed costs or increasing returns to scale
- **Multiple sectors:** Different technologies across sectors

Policy implications:

- Multiple steady states $\rightarrow$ History matters
- Policy interventions can switch between steady states
- "Big push" theories of development

Two Approaches to Solving Dynamic Models

The Question: How do we numerically solve the Ramsey model to find optimal consumption and capital paths?

Two Main Approaches:

1. **Value Function Iteration (VFI)**
   - Solves Bellman equation recursively
   - Finds policy function $c = g(k)$
2. **Shooting Algorithm**
   - Solves differential equations with boundary conditions
   - Finds specific trajectory $\{c(t), k(t)\}$

Value Function Iteration for Ramsey Model

Algorithm:

1. **Discretize** capital grid: $k \in [k_{\min}, k_{\max}]$ with N points
2. **Initial guess:** $V^0(k_i)$ for all grid points
3. **Iterate:** For $n = 0, 1, 2, \ldots$

$$V^{n+1}(k_i) = \max_c \left\{ u(c) + \beta V^n((1-\delta)k_i + k_i f'(k_i) - c) \right\}$$

   subject to: $0 \leq c \leq f(k_i) + (1-\delta)k_i$

4. **Store policy:** $c^{n+1}(k_i)$ and $k^{n+1}(k_i) = (1-\delta)k_i + k_i f'(k_i) - c^{n+1}(k_i)$
5. **Check convergence:** $\|V^{n+1} - V^n\| <$ tolerance

Implementation notes:

- Use interpolation for off-grid capital values
- Ensure $k_{\max}$ is large enough to cover relevant range
- Good initial guess speeds convergence significantly

Complete Algorithm

Algorithm 1 Ramsey VFI

Initialize: Grid $\mathcal{K} = \{k_1, k_2, \ldots, k_J\}$, Guess $V^0(k_j)$, set tolerance
Set $n = 0$
repeat
    Calculate consumption at k_j and each k_j': $c = max(\quad , 0)$
    Find Maximum: $[V^1(j), max\_idx] = max(log(c) + \beta V^0)$
    Store Asset Policy: $g_k(j) = \|(max\_idx)$
    Store Consumption Policy: $g_c(j) =$
    Check convergence: $max(|V^1 - V^0|) < \text{tol}$
    Update: $V^0 = V^1$
    $n = n + 1$
until convergence
Return: value functions, policy functions

Shooting Algorithm

Alternative approach: Exploit saddle path structure

Algorithm:

1. **Given:** Initial capital k_0
2. **Guess:** Initial consumption c_0
3. **Simulate:** Forward using Euler equation and capital evolution
4. **Check:** Does path converge to (k^*, c^*)?
5. **If not:** Adjust c_0 guess and repeat
6. **Continue:** Until convergence criterion met

Advantages:

- Very accurate along optimal path
- No grid discretization error
- Fast once you find the right initial consumption

Disadvantages:

- Only gives policy for specific k_0
- Sensitive to numerical precision
- Requires good initial guess for c_0

When to Use Value Function Iteration

VFI is Better When:

1. **Multiple initial conditions**
   - Need solutions for many different k_0
   - Policy function gives instant answers
2. **Policy analysis & comparative statics**
   - Complete characterization of optimal behavior
   - Easy to see parameter effects
3. **Stochastic models**
   - Naturally handles uncertainty
   - State-dependent policies $g(k, z)$
4. **Discrete time problems**
   - VFI is the natural approach

When to Use Shooting Algorithm

Shooting is Better When:

1. **High-dimensional state space**
   - VFI suffers from curse of dimensionality
   - Shooting scales linearly
2. **Continuous time models**
   - Natural for differential equations
   - No time discretization error
3. **High accuracy requirements**
   - Machine precision possible
   - VFI limited by grid resolution
4. **Memory constraints**
   - Minimal memory usage
   - No large arrays to store

Key Takeaways

Equilibrium concepts:

- **Sequential markets:** Period-by-period trading with perfect foresight
- **Recursive equilibrium:** State-dependent prices and policies
- **Equivalence:** Both approaches yield same allocations in representative agent models

Steady state analysis:

- **Definition:** Equilibrium with constant variables over time
- **Determination:** $f'(k^*) = \rho + \delta$ pins down k^*
- **Stability:** Saddle point with unique convergent path

Next steps: Use these concepts to analyze extensions with labor choice, government policy, and uncertainty.