Consumption-Savings Under Uncertainty

Christine Braun

University of Warwick EC9A2

Overview

So far we have discussed consumption-saving decisions under **certainty**

- partial equilibrium - exogenous interest rate r and income y
- general equilibrium - endogenous the interest rate r
- general equilibrium with labor choice $\rightarrow$ intertemporal and Intratemporal tradeoffs

Moving forward we will discuss consumption-saving decisions under **uncertainty**

- **Today:** partial equilibrium with exogenously varying interest rate r and income y
- general equilibrium - with aggregate uncertainty
- touch on idiosyncratic uncertainty

From Certainty to Uncertainty: What Changes?

Key Question: How does uncertainty about returns and income change optimal consumption and saving compared to perfect foresight?

Consumption-savings with certainty:

- Known constant return r on savings
- Predictable income stream
- Smooth consumption path (Euler equation)
- Capital converges to steady state

Consumption-savings with uncertainty:

- **Today:** Uncertain returns r_t on savings
- Uncertain labor income y_t
- Precautionary saving motives emerge
- Consumption and savings follow stochastic processes

The Economic Environment

Representative Agent: Maximizes expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where:

- $\beta \in (0, 1)$: discount factor
- $u(c_t)$: instantaneous utility function
- c_t: consumption at time t
- E_0: expectation operator conditional on information available at time 0

Standard Assumptions on Utility:

- $u'(c) > 0$, $u''(c) < 0$ (diminishing marginal utility)
- Inada conditions: $u'(0) = \infty$, $u'(\infty) = 0$

The Agent's Budget Constraint

$$c_t + k_{t+1} = (1 + r_t)k_t + y_t$$

where:

- k_t: capital stock (wealth/savings) at beginning of period t
- r_t: stochastic return on capital in period t
- y_t: stochastic labor income in period t
- c_t: consumption in period t
- k_{t+1}: savings carried into next period

Economic Interpretation:

- Agent starts period with wealth k_t
- Earns return $(1 + r_t)k_t$ on invested wealth
- Receives labor income y_t
- Allocates total resources between consumption c_t and saving k_{t+1}

Stochastic Processes

Two Sources of Uncertainty:

1. Stochastic Returns:

$$\log(1 + r_t) = \rho_r \log(1 + r_{t-1}) + \varepsilon_{r,t}$$

where $\varepsilon_{r,t} \sim N(0, \sigma_r^2)$ and $|\rho_r| < 1$

2. Stochastic Labor Income:

$$\log y_t = \rho_y \log y_{t-1} + \varepsilon_{y,t}$$

where $\varepsilon_{y,t} \sim N(0, \sigma_y^2)$ and $|\rho_y| < 1$

Assumptions:

- Both processes are AR(1) for tractability
- Shocks can be correlated: $\text{Cov}(\varepsilon_{r,t}, \varepsilon_{y,t}) = \sigma_{ry}$
- Processes are stationary (mean-reverting)

What Does E_0 Mean?

In our objective function:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

E_0 is the **expectation operator conditional on time-0 information**.

Formally:

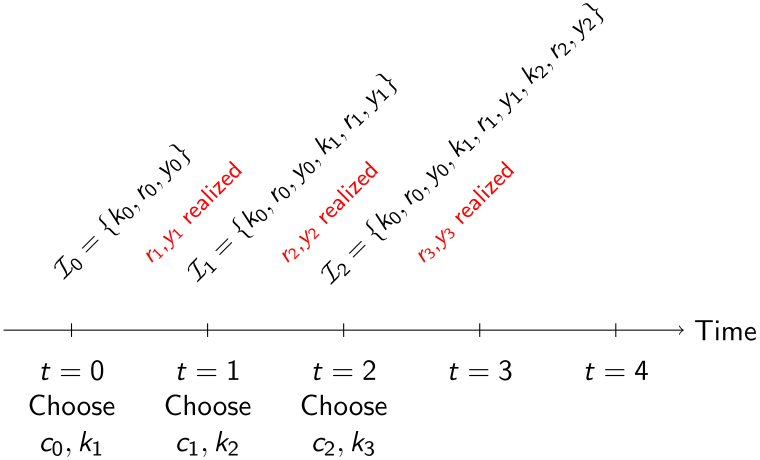
$$E_0[\cdot] = E[\cdot | \mathcal{I}_0]$$

where $\mathcal{I}_0$ is the **information set** at time 0.

What's in $\mathcal{I}_0$?

- Initial capital stock: k_0
- Initial shock: r_0 and y_0
- Knowledge of stochastic processes: ρ_r, σ_r^2, ρ_y, σ_y^2 and σ_{ry}
- All model parameters (β, etc.)

Evolution of Information Sets: Timeline



Key Point: At each date t, agent knows current and all past shocks, but future shocks are uncertain.

Information Sets Over Time

Information evolves as shocks are realized:

At time 0: $\mathcal{I}_0 = \{k_0, r_0, y_0, \text{model parameters}\}$
At time 1: $\mathcal{I}_1 = \{k_0, r_0, y_0, k_1, r_1, y_1, \text{model parameters}\}$
At time 2: $\mathcal{I}_2 = \{k_0, r_0, y_0, k_1, r_1, y_1, k_2, r_2, y_2, \text{model parameters}\}$
And so on...

Key Point: $\mathcal{I}_t \supseteq \mathcal{I}_{t-1}$ (information never decreases)

Notation:

$$E_t[\cdot] = E[\cdot|\mathcal{I}_t]$$

This is the expectation conditional on all information available at time t.

The Challenge: From Sequential to Recursive

Sequential Problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to budget constraints for all t.

How can we get a recursive problem: Through time-separability and the law of iterated expectations.

First we need to determine the control and state variables

- Control: consumption c_t and next period capital k_{t+1}
- State: ?

State Variables: What and Why?

Definition: a state variables is a variable whose value:

1. Carries over from period to period (persistence)
2. Cannot be chosen freely in the current period (predetermined)
3. Summarizes relevant history for decision-making
4. Affects future constraints and opportunities

Our State Variables: (k_t, r_t, y_t)

Key Property - Markov: Given (k_t, r_t, y_t), the entire history $(k_0, r_0, y_0, \ldots, k_{t-1}, r_{t-1}, y_{t-1})$ is irrelevant for optimal decisions.

Why Each State Variable is Necessary

1. **Capital/Wealth (k_t):**
   - **Carries over to** $t+1$
   - **Predetermined at time** t (chosen at $t-1$)
   - **Determines return income:** $(1+r_t)k_t$
   - **Summarizes all past decisions:** Accumulated result of past consumption/saving

2. **Current Return (r_t):**
   - **Affects current resources:** $(1+r_t)k_t$ depends on r_t
   - **Persistent process and carry over:** r_t predicts r_{t+1} (AR(1) with $\rho_r \neq 0$)
   - **Summarizes history:** Markov property

3. **Current Income (y_t):**
   - **Direct budget impact:** Available resources for consumption/saving
   - **Persistent process and carry over:** y_t predicts y_{t+1} (AR(1) with $\rho_y \neq 0$)
   - **Summarizes history:** Markov property

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How can we get a recursive problem: Through time-separability and the law of iterated expectations.

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- State: (k_1, r_t, y_t)

Law of Iterated Expectations

Statement: For any random variable X and information sets $\mathcal{I}_s \subseteq \mathcal{I}_t$:

$$E[X|\mathcal{I}_s] = E[E[X|\mathcal{I}_t]|\mathcal{I}_s]$$

Special Case (Tower Property): When $s < t$:

$$E_s[X] = E_s[E_t[X]]$$

Intuitive Interpretation:

- Today's expectation of X equals today's expectation of tomorrow's expectation of X
- Information revealed between s and t doesn't change the s-period expectation on average

Time-Separable Utility Function

Our utility specification:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Key Properties:

1. **Additively separable**: Total utility is sum of period utilities
2. **No direct cross-period effects**: u_t depends only on c_t
3. **Constant discounting**: Same β in each period

What this rules out:

- Habit formation: $u(c_t, c_{t-1})$
- Durability: $u(c_t + \alpha c_{t-1})$
- Time-varying discounting: $\sum_{t=0}^{\infty} \beta_t u(c_t)$

Step 1: Separate Current Period

Start with:

$$V(k_0, r_0, y_0) = \max_{\{c_t, k_{t+1}\}_0^\infty} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Separate first period:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = E_0[u(c_0)] + E_0 \sum_{t=1}^{\infty} \beta^t u(c_t)$$

Since c_0 is chosen at time 0 (no uncertainty):

$$= u(c_0) + E_0 \sum_{t=1}^{\infty} \beta^t u(c_t)$$

Factor out β:

$$= u(c_0) + \beta E_0 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$

Step 2: Apply Law of Iterated Expectations

We have:

$$u(c_0) + \beta E_0 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$$

Key insight: Decisions from $t = 1$ onward will be made optimally given info at $t = 1$.

Apply law of iterated expectations:

$$E_0 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) = E_0 \left[E_1 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right]$$

But the inner expectation is just the value function at $t = 1$:

$$E_1 \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) = V(k_1, r_1, y_1)$$

Therefore:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = u(c_0) + \beta E_0[V(k_1, r_1, y_1)]$$

Step 3: The Recursive Structure Emerges

From the previous slide:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) = u(c_0) + \beta E_0[V(k_1, r_1, y_1)]$$

The agent's problem becomes:

$$\max_{c_0, k_1} \{u(c_0) + \beta E_0[V(k_1, r_1, y_1)]\}$$

subject to the time-0 budget constraint.

Key observation: This has the same structure in every period

Why the Same Function in Every Period?

Question: Why is $V(\cdot, \cdot)$ the same function in all periods?
Answer: Two key assumptions ensure this:

1. **Stationarity:**
   - Stochastic process parameters (ρ_r, σ_r^2, ρ_y,σ_y^2, and σ_{ry}) don't change over time
   - Preference parameters (β, $u(\cdot)$) don't change over time

2. **Markov Property:**
   - Future state $(k_{t+1}, r_{t+1}, y_{t+1})$ depends only on current state (k_t, r_t, y_t)
   - No additional state variables carry information about the future
   - History beyond current state is irrelevant for optimal decisions

The Markov Property

Markov Property for our model:
The transition probability satisfies:

$$\Pr(r_{t+1}, y_{t+1} | r_t, y_t r_{t-1}, y_{t-1}, \ldots, r_0, y_0) = \Pr(r_{t+1}, y_{t+1} | r_t, y_t)$$

Implication: State (k_t, r_t, y_t) is **sufficient statistic** for making optimal decisions.

This enables us to write:

$$V(k_t, r_t, y_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta E[V(k_{t+1}, r_{t+1}, y_{t+1}) | r_t, y_t]\}$$

Note: In the future state $(k_{t+1}, r_{t+1}, y_{t+1})$ only r_{t+1} and y_{t+1} are uncertain, k_{t+1} is already chosen by the agent, so we condition only on r_t and y_t

Expectations Conditional on Current State

In the Bellman equation: (doping the time subscript)

$$V(k, r, y) = \max_{c,k'} \{u(c) + \beta E[V(k', r', y')|r, y]\}$$

What does $E[\cdot|r, y]$ mean?

- Expectation over next period's interest rate r' and income y'
- Conditional on current interest rate r and income y
- Using the known transition probabilities

For our AR(1) process:

$$E[V(k', r', y')|A_L] = \int_{r,y} V(k', r, y)dF(r, y)$$

where $F(r, y)$ is the joint cdf of r and y.

The Bellman Equation

The agent's problem can be written recursively as:

$$V(k, r, y) = \max_{c,k'} \left\{ u(c) + \beta E[V(k', r', y') | r, y] \right\}$$

subject to:

$$\begin{aligned} c + k' &= (1 + r)k + y && (1) \\ k' &\geq 0 \quad \text{no borrowing} && (2) \\ (r', y') &\sim \text{joint stochastic process} && (3) \end{aligned}$$

State Variables: (k, r, y) - wealth, current return, current income
Control Variables: c, k' - consumption and next-period wealth

First-Order Conditions

FOC for consumption:

$$u'(c) = \beta E[V'(k', r', y')|r, y]$$

Envelope condition:

$$V'(k, r, y) = (1 + r)u'(c)$$

Combining these yields the **stochastic Euler equation:**

$$u'(c) = \beta E[(1 + r')u'(c')|r, y]$$

If the borrowing constraint binds:

$$u'(c) > \beta E[(1 + r')u'(c')|r, y]$$

The Stochastic Euler Equation: Key Insights

Using time subscripts

$$u'(c_t) = \beta E_t[(1 + r_{t+1})u'(c_{t+1})]$$

Two Sources of Uncertainty:

1. **Return uncertainty**: $(1 + r_{t+1})$ is random
2. **Consumption uncertainty**: c_{t+1} is random (depends on y_{t+1})

Implications:

- Can't simply use expected values: $E[XY] \neq E[X]E[Y]$ in general
- Covariance between returns and marginal utility matters
- Jensen's inequality effects from convex marginal utility

Decomposing the Right-Hand Side

We can rewrite the expectation as:

$$\beta E_t[(1+r_{t+1})u'(c_{t+1})] =$$

$$\beta E_t[1+r_{t+1}] \cdot E_t[u'(c_{t+1})] + \beta \mathrm{Cov}_t[(1+r_{t+1}), u'(c_{t+1})]$$

This gives us **three economic effects**:

1. **Expected Return Effect**: $\beta E_t[1+r_{t+1}] \cdot E_t[u'(c_{t+1})]$
2. **Precautionary Saving Effect**: $E_t[u'(c_{t+1})] \neq u'(E_t[c_{t+1}])$
3. **Risk Premium Effect**: $\mathrm{Cov}_t[(1+r_{t+1}), u'(c_{t+1})]$

Effect 1: Expected Return Effect

$$\beta E_t[1 + r_{t+1}] \cdot E_t[u'(c_{t+1})]$$

Economic interpretation:

- Higher expected returns make saving more attractive
- Similar to deterministic case but uses expected values
- Standard intertemporal substitution effect

Example:

- If $E_t[r_{t+1}]$ increases (e.g., Fed raises interest rates)
- Agent finds it optimal to save more, consume less today
- Future consumption becomes relatively cheaper

Policy implication: Monetary policy affects consumption through expected return channel.

Effect 2: Precautionary Saving Effect

By Jensen's inequality, when $u'''(c) > 0$ (convex marginal utility):

$$E_t[u'(c_{t+1})] > u'(E_t[c_{t+1}])$$

- Uncertainty about future consumption raises expected marginal utility
- Makes saving more attractive even if expected consumption is unchanged

Result: Uncertainty about future consumption $\Rightarrow$ save more today!

Effect 3: Risk Premium Effect

$$\text{Cov}_t[(1 + r_{t+1}), u'(c_{t+1})]$$

The sign of this covariance determines the risk properties of the asset:

Case 1: $\text{Cov}[(1 + r_{t+1}), u'(c_{t+1})] < 0$ (Negative)

- Low returns occur when marginal utility is high (bad times)
- Asset is **risky** from consumption-smoothing perspective
- Agent demands **risk premium** (higher expected return)
- **Example**: Stocks that crash during recessions

Case 2: $\text{Cov}[(1 + r_{t+1}), u'(c_{t+1})] > 0$ (Positive)

- High returns occur when marginal utility is high (bad times)
- Asset provides **insurance** against consumption risk
- Agent accepts **lower expected return**
- **Example**: Safe bonds that appreciate during recessions

Summary: Three Forces in the Stochastic Euler Equation

The stochastic Euler equation: $u'(c_t) = \beta E_t[(1 + r_{t+1})u'(c_{t+1})]$ captures **three fundamental economic forces**:

1. **Expected Return Effect (Standard intertemporal substitution)**
   - Trade off consumption today vs. tomorrow
   - Driven by expected returns: $E_t[1 + r_{t+1}]$
2. **Precautionary saving**
   - Uncertainty about future consumption
   - Jensen's inequality: $E_t[u'(c_{t+1})] > u'(E_t[c_{t+1}])$
3. **Risk premium effects**
   - Covariance between returns and marginal utility
   - Determines which assets are "safe" vs. "risky"

Special Case 1: Only Income Risk

Setup: Constant returns r, uncertain income y_t
Euler equation becomes:

$$u'(c_t) = \beta(1 + r)E_t[u'(c_{t+1})]$$

Key insight: Pure precautionary effect

- $E_t[u'(c_{t+1})] > u'(E_t[c_{t+1}])$ when income is uncertain
- Agent saves more than in deterministic case
- Building buffer stock against income shocks

Economic interpretation:

- Higher expected marginal utility makes saving attractive
- Wealth serves as self-insurance against income volatility
- Stronger effect with higher risk aversion

Special Case 2: Only Return Risk

Setup: Uncertain returns r_t, constant income $\bar{y}$

Euler equation:

$$u'(c_t) = \beta E_t[(1 + r_{t+1})u'(c_{t+1})]$$

Economic interpretation:

- Agents face **portfolio risk** - returns on savings are uncertain
- Higher return volatility $\Rightarrow$ more precautionary saving

Two competing effects:

1. **Higher expected returns**: Encourages more saving
2. **Return volatility**: May discourage saving (risk premium effect)

Net effect depends on:

- Higher risk aversion $\Rightarrow$ volatility effect dominates
- **Wealth level**: Rich agents can better tolerate return risk

How do we solve these models?

Simplify: Lets consider only special case 1: stochastic income

$$r_t = r \quad \text{and} \quad y_t \in \{y_L, y_H\}$$

with transition matrix P.

No closed-form solution because:

- Nonlinear Euler equation with expectations
- State-dependent policy functions
- Stochastic income process creates complex dynamics

Use Value function iteration (VFI) to get:

- **Value Function**: $V(k, y)$
- **Consumption policy function**: $g_c(k, y)$
- **Capital policy function**: $g_k(k, y)$

Basic idea: Start with guess for value function, iterate until convergence using Bellman operator.

Discretize State Space

Income space: Already discrete

- Income grid: $\{y_L, y_H\}$
- Transition matrix: P

Capital grid:

- Choose bounds: $k \in [k_{\min}, k_{\max}]$
- $k_{\min} = 0$ (no borrowing)
- $k_{\max}$: Large enough that never reached in equilibrium
- Grid points: $\{k_1, k_2, \ldots, k_{N_k}\}$

Total state space: $N_k \times 2$ grid points

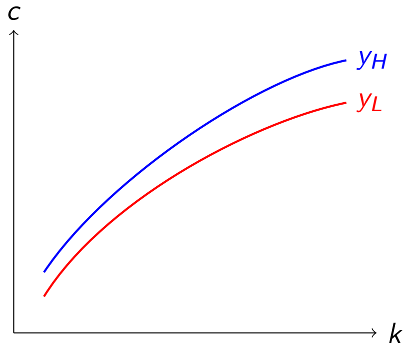
VFI Algorithm: Simplified Version

Algorithm 1 Value Function Iteration - Two State

```
Initialize: V^0(k_i, y_L), V^0(k_i, y_H) for all i
repeat
    for i = 1 to N_k do
        // Low income state
        V^{n+1}(k_i, y_L) = max_{k'} {u(c) + β[p_{LL} V^n(k', y_L) + p_{LH} V^n(k', y_H)]}
        where c = (1 + r)k_i + y_L − k'
        Store: k'_{i,L} = g_k(k_i, y_L)
        // High income state
        V^{n+1}(k_i, y_H) = max_{k'} {u(c) + β[p_{HL} V^n(k', y_L) + p_{HH} V^n(k', y_H)]}
        where c = (1 + r)k_i + y_H − k'
        Store: k'_{i,H} = g_k(k_i, y_H)
    end for
    n = n + 1
until max_i |V^{n+1}(k_i, y_j) − V^n(k_i, y_j)| < ϵ for j ∈ {L, H}
```

Consumption Policy Function

Optimal consumption: $c = g_c(k, y)$

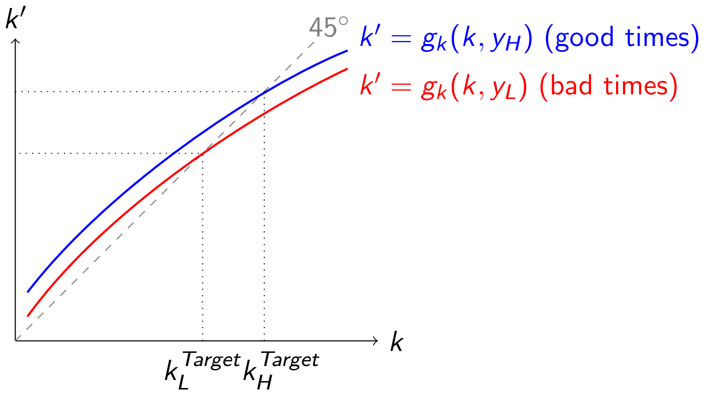


Key Properties:

- **Increasing in wealth**: $\frac{\partial g^c}{\partial k} > 0$
- **Concave in wealth**: Diminishing marginal propensity to consume

Savings Policy Function

Optimal savings: $k' = g_k(k, y)$



Key insights:

- **State-dependent targets**: Different wealth targets for different income states
- **Buffer stock behavior**: Build up wealth in good times, run down in bad times

Buffer Stock Behavior

Key Insight: Wealth serves dual purpose:

1. **Standard Ramsey role**: Smooth consumption over time
2. **Insurance role**: Buffer against income/return shocks

Implications:

- **Target wealth level**: Higher than deterministic steady state
- **State-dependent behavior**:
  - After bad shocks: Cut consumption, rebuild wealth
  - After good shocks: Increase consumption, moderate wealth growth
- **Incomplete consumption smoothing**: Can't fully insure against all risks

Micro Evidence: Consistent with observed household behavior:

- Higher saving rates for uncertain income groups
- "Hand-to-mouth" behavior for low-wealth households

Effect of Risk Aversion (θ)

Higher Risk Aversion:

- **Stronger precautionary motive**: More saving for given uncertainty
- **Lower consumption**: For any wealth level, consume less
- **Higher target wealth**: Build larger buffer stocks
- **Less sensitivity to shocks**: Smoother consumption profile

Prudence measures how much an agent dislikes "downside risk" - the tendency to take precautionary actions when facing uncertainty.

$$-\frac{cu'''(c)}{u''(c)}$$

Economic interpretation: How much the marginal utility curve "bends" (convexity of marginal utility). Higher $\theta \Rightarrow$ higher prudence $\Rightarrow$ more precautionary saving.

Effect of Income Volatility

Higher Income Uncertainty:

- **Unambiguous increase in saving**: Pure precautionary effect
- **Higher target wealth**: Need bigger buffer for income shocks
- **More volatile consumption**: Despite higher saving, consumption still fluctuates

For AR(1) process, **Income Persistence (ρ_y) matters:**

- **Temporary shocks** (ρ_y low): Smooth through saving/dis-saving
- **Persistent shocks** (ρ_y high): Must adjust consumption more
- **Permanent shocks** ($\rho_y = 1$): Consumption tracks income closely

Implication: Nature of income risk (temporary vs. permanent) crucially affects optimal consumption smoothing.

Wealth Effects vs. Substitution Effects

Model with stochastic return r

Response to a positive return shock depends on wealth level:

Low Wealth ("Poor" agents):

- **Strong wealth effect**: Higher returns $\Rightarrow$ feel richer $\Rightarrow$ consume more
- **Weak substitution effect**: Close to subsistence $\Rightarrow$ can't reduce consumption much
- **Net effect**: Consumption increases significantly with good return shocks

High Wealth ("Rich" agents):

- **Weak wealth effect**: Already wealthy $\Rightarrow$ marginal utility low
- **Strong substitution effect**: Can afford to save more when returns high
- **Net effect**: Consumption less sensitive to return shocks

Policy Implication: Monetary policy (affecting returns) has distributional consequences - affects poor more than rich.

Key Takeaways

Main Insights:

1. **Uncertainty fundamentally changes saving behavior** - even with perfect capital markets
2. **Precautionary saving emerges** when marginal utility is convex ($u''' > 0$)
3. **Wealth serves as insurance** - buffer stock behavior
4. **Risk aversion amplifies** all these effects
5. **Wealth level matters** - rich and poor respond differently to shocks

Connection to Broader Literature:

- Foundation for heterogeneous agent models
- Links to asset pricing through stochastic discount factor
- Basis for understanding incomplete markets economies