Consumption-Savings Under Uncertainty Cont.

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Overview

Last Time:

- partial equilibrium with exogenously varying interest rate r and income y
- VFI for model with stochastic income
- risk aversion and capital targets

Today:

- steady state in stochastic models
- solving for a steady state
- determining equilibrium interest rate

Two Interpretations of the Stochastic Partial Equilibrium Model

Representative Agent Model

- One agent over infinite time
- Faces idiosyncratic shocks
- Makes optimal decisions

Steady state represents

- Agent's long-run time averages
- Ergodic distribution over lifetime

Economic focus

- Lifetime welfare analysis
- Individual consumption smoothing
- Time series predictions

Heterogeneous Agents Model

- Many agents at single point in time
- Each in different states (k, y)
- All solve same problem

Steady state represents

- Cross-sectional distribution
- Fraction of population in each state

Economic focus

- Inequality and redistribution
- Distributional effects of policy
- Cross-sectional patterns

Two Interpretations of the Stochastic Partial Equilibrium Model

Key insight: Both interpretations are **mathematically identical** due to the ergodic theorem:

$$\text{Time averages} = \text{Cross-sectional averages}$$

Choose interpretation based on: Your research question, available data, and policy application.

VERY IMPORTANT: when we endogenize the interest rate, the distinction between representative agent and heterogeneous agent is not innocuous.

Moving forward we will use the heterogeneous agent interpretation.

Heterogeneous Agent Models in Macroeconomics

What are Heterogeneous Agent Models?

- Models with a **continuum of agents** who differ in characteristics
- Agents may differ in: wealth, income, productivity, preferences, demographics
- **Distribution of agents** becomes a state variable of the economy

Key Features:

- **Individual heterogeneity** matters for aggregate outcomes
- **Distribution dynamics** evolve endogenously over time
- **Non-linear aggregation**: average behavior $\neq$ behavior of average agent
- Often feature **incomplete markets** (limited insurance/borrowing)

Why Study Heterogeneous Agent Models?

- **Empirical Motivation**: Substantial wealth inequality in the data
  - Top 1% holds $\sim$40% of total wealth (US)
  - Bottom 50% holds $\sim$2% of total wealth
- **Theoretical Motivation**: Representative agent models miss important mechanisms
  - Precautionary saving motives
  - Distributional effects of policy
  - Aggregate implications of individual heterogeneity
- **Policy Relevance**: Distributional consequences matter for welfare analysis

What Are They Used For?

1. **Wealth and Income Inequality**
   - Explain observed wealth distributions
   - Study drivers of inequality: returns to capital, labor market frictions, etc.
   - Analyze distributional effects of policies (taxes, transfers, regulations)
2. **Consumption and Savings Behavior**
   - **Precautionary saving**: agents save to self-insure against income risk
   - Explain "excess sensitivity" of consumption to income changes
   - Buffer-stock saving and target wealth levels
3. **Asset Pricing Puzzles**
   - Low real interest rates despite high time preference rates
   - Limited participation in stock markets
   - Risk premiums that vary with wealth distribution

What Are They Used For?

4. **Business Cycle Analysis**
   - How do recessions affect different groups differently?
   - Role of wealth distribution in propagating shocks
   - **Amplification effects**: heterogeneity can magnify aggregate fluctuations
5. **Policy Analysis**
   - **Distributional consequences** of monetary policy, fiscal policy
   - Design of optimal tax and transfer systems
   - Social insurance programs (unemployment, health, pensions)
   - Financial regulation and consumer protection
6. **Labor Market Dynamics**
   - Job search with heterogeneous workers and firms
   - Unemployment duration and wage dispersion
   - Effects of minimum wages, UI benefits, job training programs

Key Model Classes

1. **Incomplete Markets Models**
   - **Bewley (1986)**, **Huggett (1993)**, **Aiyagari (1994)**
   - Idiosyncratic income risk, limited borrowing, precautionary saving
2. **Life-Cycle Models**
   - Heterogeneity in age, earnings profiles, mortality risk
   - Social Security, pensions, intergenerational transfers
3. **Search and Matching Models**
   - Labor market frictions, unemployment, job-to-job transitions
   - Wage posting, bargaining, firm heterogeneity
4. **HANK Models (Heterogeneous Agent New Keynesian)**
   - Combine heterogeneous agents with nominal rigidities
   - Monetary policy transmission through distributional channels

The Huggett (1993) Model

Key Innovation: Heterogeneous agents with incomplete markets

Environment:

- Agents face idiosyncratic labor productivity shocks
- Cannot fully insure against these shocks
- Only asset available: risk-free bonds

Central Questions:

- How much precautionary saving occurs?
- What is the equilibrium interest rate?
- What wealth distribution emerges?

Note: This is the model we solved last lecture with stochastic income and fixed interest rate. Now we will endogenize the interest rate.

Complete Markets Benchmark

Complete Markets: Agents can trade state-contingent securities for all possible future states

Implications:

- Perfect risk sharing across agents
- Consumption independent of individual productivity realizations
- No precautionary saving motive
- Interest rate equals rate of time preference: $r = \rho$ where $\beta = \frac{1}{1+\rho}$

Incomplete Markets

Incomplete Markets: Limited set of financial instruments available

In Huggett Model:

- Only risk-free bonds available
- No state-contingent insurance
- Agents cannot perfectly smooth consumption

Implications:

- Consumption varies with individual productivity shocks
- Precautionary saving emerges as self-insurance
- Wealth distribution becomes non-degenerate
- Interest rate differs from time preference rate

Environment

Time: $t = 0, 1, 2, \ldots$ (discrete, infinite horizon)

Agents: Continuum of measure 1, indexed by $i \in [0, 1]$

Preferences:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t})$$

where $0 < \beta < 1$ and $u(\cdot)$ is strictly concave, increasing

Productivity Process: Each agent i receives productivity shock $y_{i,t}$

- $y_{i,t} \in \mathcal{Y} = \{y_1, y_2, \ldots, y_N\}$ (finite support)
- Markov process with transition matrix P
- $P_{jk} = \Pr(y_{t+1} = y_k | y_= y_j)$
- Stationary distribution π where $\pi = \pi P$

Assets and Budget Constraint

Assets: Risk-free bonds $a_{i,t}$ with gross return $1 + r$

Budget Constraint:

$$c_{i,t} + a_{i,t+1} = y_{i,t} + (1 + r)a_{i,t}$$

Borrowing Constraint: $a_{i,t+1} \geq -\phi$ for some $\phi \geq 0$

- $\phi = 0 \Rightarrow$ No borrowing
- Natural borrowing limit: $\bar{\phi} = \sum(1 + r)^{-(t+1)} y_1$ (typically non-binding)
- $0 < \phi < \bar{\phi} \Rightarrow$ limited borrowing

No-Ponzi Condition: Prevents agents from borrowing indefinitely to finance consumption. "You can't have negative wealth forever in present value terms."

$$\liminf_{T \to \infty} \mathbb{E}_t[R^{-T} a_{i,t+T}] \geq 0$$

Bellman Equation

Agent's problem:

$$V(a,y) = \max_{c,a'} \left\{ u(c) + \beta E[V(a',y')|y] \right\}$$

subject to:

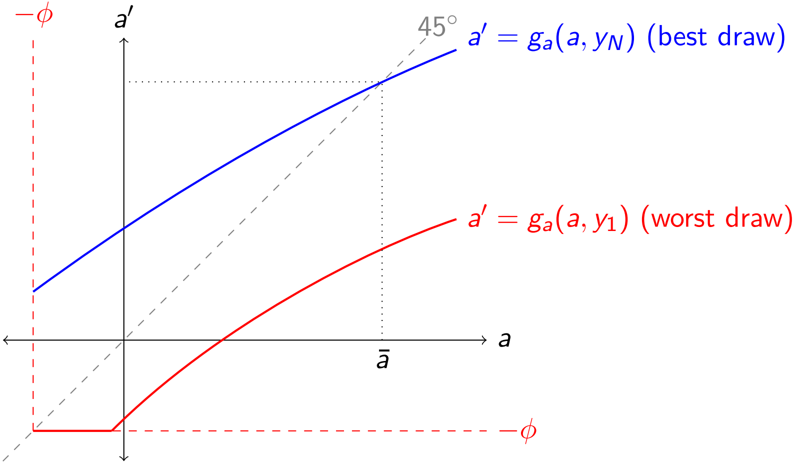
$$c + a' = (1+r)a + y$$
$$a' \geq -\phi$$
$$c \geq 0$$
$$y \sim \text{stochastic process}$$

Key features:

- **Exogenous return**: r is constant (partial equilibrium)
- **Stochastic income**: y_t follows a Markov process
- **Borrowing**: agents can borrow up to $-\phi$

Solution yields: Policy functions $c = g_c(a,y)$ and $a' = g_a(a,y)$

Asset Policy Functions



Asset Policy Functions

Unconstrained: (blue line)

- agent will dissave if $a > \bar{a}$
- agen will save if $a < \bar{a}$
- $\bar{a}$ is the target wealth level when $y = y_N$

Constrained: (red line)

- agent is always dissaving, regardless of asset level
- flat part of policy function is when the agent is borrowing constrained
- $-\phi$ is the target wealth level when $y = y_1$

Key Feature: Any agent that start with initial endowment $a \in [-\phi, \bar{a}]$ will stay within this set of assets. Let $\mathcal{S} = [-\phi, \bar{a}] \times \mathcal{Y}$, then

$$g_a : \mathcal{S} \to [-\phi, \bar{a}]$$

Deterministic vs. Stochastic Steady States

Deterministic steady state:

- **Fixed point**: $k_t = k^* =$ constant for all t
- **No randomness**: $c_t = c^* =$ constant for all t

Stochastic steady state:

- **Distributional equilibrium**: Joint distribution of (a_t, y_t) is time-invariant
- **Randomness**: a_t, c_t still fluctuate
- **Statistical stability**: All moments constant over time

Key insight: In stochastic models, we have a stationary distribution, not a fixed point.

Definition: Stochastic Steady State

Formal definition: A stochastic steady state is a probability measure $\mu^*(a, y)$ such that:

$$\mu^*(a', y') = \int\int_{a,y} \mu^*(a, y) \cdot Q\big((a, y) \to (a', y')\big)\ da\ dy$$

where Q is the transition function defined by optimal policy and income process.

In plain English:

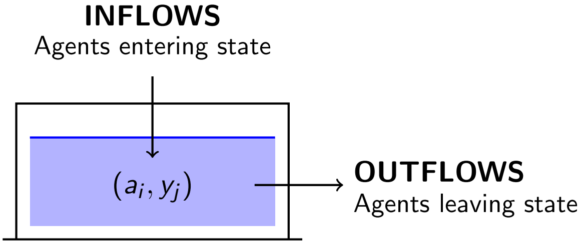
- The **cross-sectional distribution** of assets and income is constant over time
- Individuals move between states, but the **fraction** in each state stays the same
- All **aggregate statistics** (means, variances, correlations) are time-invariant

Properties:

- $E[a_t] = E[a_{t+1}]$ (constant mean assets)
- $\text{Var}[a_t] = \text{Var}[a_{t+1}]$ (constant asset inequality)
- All higher moments are also constant

The Bathtub Analogy

Steady state = Constant water level in a bathtub



Steady State: Inflow Rate = Outflow Rate

For any state (a_i, y_j):

- **Inflows**: Agents arriving from other states
- **Outflows**: Agents leaving for other states
- **Steady state**: Flow in = Flow out

If inflows $\neq$ outflows $\rightarrow$ distribution changes over time $\rightarrow$ not steady state!

The Steady State Condition

Example: Lets discretize the space of capital, $a \in \{a_1, a_2, ..., a_N\}$. The income space and probabilities are defined by the discrete Markov process.

For every state (a_i, y_j)**, we need:** inflows $=$ outflows

Inflows to (a_i, y_j)**:**

$$\sum_{a_\ell, y_m} \mu^*(a_\ell, y_m) \cdot \mathbb{I}[g_a(a_\ell, y_m) = a_i] \cdot P_{mj}$$

Outflows from (a_i, y_j)**:** $\mu^*(a_i, y_j)$
Steady state condition:

$$\sum_{a_\ell, y_m} \mu^*(a_\ell, y_m) \cdot \underbrace{\mathbb{I}[g_a(a_\ell, y_m) = k_i] \cdot P_{mj}}_{Q} = \mu^*(a_i, y_j)$$

This must hold for **every** state (a_i, y_j) simultaneously.

Inflows: Who Can Enter State (a_i, y_j)?

"How many agents will be at (a_i, y_j) next period?"

Inflow formula:

$$\sum_{a_\ell, y_m} \mu^*(a_\ell, y_m) \cdot \mathbb{I}[g_a(a_\ell, y_m) = a_i] \cdot P_{mj}$$

Breaking it down:

- $\mu^*(a_\ell, a_m)$: Fraction of agents currently at state (a_ℓ, y_m)
- $\mathbb{I}[g_a(a_\ell, y_m) = a_i]$: Indicator $= 1$ if agents at (a_ℓ, y_m) optimally choose assets a_i
- P_{mj}: Probability that income transitions from y_m to y_j

Economic interpretation:

- Sum over **all possible previous states** (a_ℓ, y_m)
- Include those where agents **choose** $a' = a_i$ (optimal decision)
- Weight by probability of income transition $y_m \to y_j$ (random)

Small caveat: for discrete spaces this formula includes agents that stay, but these agents show up on both sides of our steady state condition.

Outflows: Who Leaves State (a_i, y_j)?

Outflow formula:

$$\mu^*(a_i, y_j)$$

Why is this so simple?

- All agents currently at (a_i, y_j) will be in different states next period
- State variables change every period due to optimal choices and income shocks

Where do they go?

- **Asset choice**: $a' = g_a(a_i, y_j)$ (deterministic, optimal)
- **Income transition**: $y_j \to y_\ell$ with probability $P_{j\ell}$ (random)
- **New states**: $(g_a(a_i, y_j), y_\ell)$ for various y_ℓ

The outflow splits among different destination states, but total outflow is always $\mu^*(a_i, y_j)$.

Small caveat: for discrete spaces this formula includes agents that stay, but these agents show up on both sides of our steady state condition.

What Happens If Inflows $\neq$ Outflows?

Case 1: Inflows $>$ Outflows

- State (a_i, y_j) accumulates agents over time
- $\mu(a_i, y_j)$ increases each period
- Distribution is not stationary

Case 2: Inflows $<$ Outflows

- State (a_i, y_j) loses agents over time
- $\mu(a_i, y_j)$ decreases each period
- Distribution is not stationary

Only when Inflows $=$ Outflows:

- Population at each state remains constant
- $\mu(a_i, y_j)$ doesn't change over time
- Distribution is stationary $\rightarrow$ steady state!

Key Properties of Steady State Distribution

1. **Ergodicity:**
   - Any individual agent will eventually visit all possible states
   - Long-run time averages = cross-sectional averages
   - Independent of initial conditions (for most starting points)

2. **Unique stationary distribution:**
   - Under standard conditions, there's exactly one μ^*
   - Follows from contraction mapping properties

3. **Buffer stock behavior:**
   - **Mean assets**: $E[a] > a^*_{\text{deterministic}}$ (precautionary saving)
   - **Wealth dispersion**: $\sigma_a > 0$ (permanent inequality)
   - **Consumption smoothing**: $\sigma_c < \sigma_y$ (but imperfect)

4. **Income-asset correlation:**
   - $\text{Cov}(a, y) > 0$ typically (rich tend to have higher income)
   - But correlation is imperfect due to consumption smoothing

Two-Step Solution Procedure For solving model

Step 1: Solve for Policy Functions

- Use value function iteration to find $g_a(a, y)$ and $g_c(a, y)$
- This gives us the optimal decision rules
- We covered this in previous lectures

Step 2: Find Steady State Distribution

- Given policy functions, simulate the distribution forward in time
- Start with some initial distribution $\mu^0(a, y)$
- Iterate using transition probabilities until convergence

Key insight: Steps are sequential - policy functions don't depend on distribution in partial equilibrium, but distribution depends on policy functions.

In general equilibrium: Would need to iterate between both steps. Why?

Step 1 Recap: Policy Function Solution

Already covered, but quick reminder:

Algorithm 1 Value Function Iteration (Review)

```
Initialize: $V^0(a_i, y_j)$ for all $(i, j)$
repeat
    for $i = 1$ to $N_a$ do
        for $j = 1$ to $N_y$ do
            $V^{n+1}(a_i, y_j) = \max_{a'} \{u(c) + \beta \sum_\ell P_{j\ell} V^n(a', y_\ell)\}$
            where $c = (1 + r)a_i + y_j - a'$
            Store: $g_a(a_i, y_j) = \arg\max_{a'}\{\cdots\}$
        end for
    end for
until Convergence
Return: Policy functions $g_a(\cdot, \cdot)$ and $g_c(\cdot, \cdot)$
```

Output: Optimal policy functions on the grid.

Step 2: Distribution Iteration Algorithm

Algorithm 2 Steady State Distribution

Input: Policy functions $g_a(a_i, y_j)$, transition matrix P

Initialize: $\mu^0(a_i, y_j) = \frac{1}{N_a \times N_y}$ (uniform distribution)

repeat

  $\mu^{n+1}(a_i, y_j) = 0$ for all (i, j)

  for $\ell = 1$ to N_a **do**

    for $m = 1$ to N_y **do**

      $a_i = g_a(a_\ell, y_m)$

      for $j = 1$ to N_y **do**

        $\mu^{n+1}(a_i, y_j) = \mu^{n+1}(a_i, y_j) + \mu^n(a_\ell, y_m) \cdot P_{mj}$

      end for

    end for

  end for

until $\|\mu^{n+1} - \mu^n\|_1 < \epsilon$

Return: Steady state distribution μ^*

Understanding the Distribution Iteration

The core idea: Track how probability mass flows between states.

For each current state (a_ℓ, y_m):

1. Agent chooses next-period capital: $a' = g_a(a_\ell, y_m)$
2. Income transitions according to: $y_m \to y_j$ with probability P_{mj}
3. Probability mass $\mu^n(a_\ell, y_m) \times P_{mj}$ flows to state (a_i, y_j)

Line 9 in Algorithm 2:

$$\mu^{n+1}(a_i, y_j) = \sum_{\ell,m} \mu^n(a_\ell, y_m) \cdot \mathbb{I}[g_a(a_\ell, y_m) = a_i] \cdot P_{mj}$$

Convergence condition:

$$\mu^{n+1}(a_i, y_j) = \mu^n(a_i, y_j) \quad \text{for all } (i, j)$$

Convergence and Validation

Convergence criteria:

- **L1 norm**: $\|\mu^{n+1} - \mu^n\|_1 = \sum_{i,j} |\mu^{n+1}(a_i, y_j) - \mu^n(a_i, y_j)| < \epsilon$
- **Supremum norm**: $\|\mu^{n+1} - \mu^n\|_\infty = \max_{i,j} |\mu^{n+1}(a_i, y_j) - \mu^n(a_i, y_j)| < \epsilon$
- **Typical tolerance**: $\epsilon = 10^{-6}$ to 10^{-8}

Validation checks:

- **Probability conservation**: $\sum_{i,j} \mu^*(a_i, y_j) = 1$
- **Non-negativity**: $\mu^*(a_i, y_j) \geq 0$ for all (i, j)
- **Stationarity test**: Apply one more iteration, check result unchanged

Extracting Economic Statistics

First moments (means):

$$E[a] = \sum_{i,j} a_i \mu^*(a_i, y_j)$$

$$\bar{y} = \sum_{i,j} y_j \mu^*(a_i, y_j)$$

$$\bar{c} = \sum_{i,j} g_c(a_i, y_j) \mu^*(a_i, y_j)$$

Second moments (variances):

$$\sigma_a^2 = \sum_{i,j} (a_i - E[a])^2 \mu^*(a_i, y_j)$$

$$\sigma_c^2 = \sum_{i,j} (g_c(a_i, y_j) - \bar{c})^2 \mu^*(a_i, y_j)$$

Correlations:

$$\mathrm{Corr}(a, y) = \frac{\sum_{i,j} (a_i - E[a])(y_j - \bar{y}) \mu^*(a_i, y_j)}{\sigma_a \sigma_y}$$

Economic Interpretation of Moments

Mean assets $E[a]$:

- Compare to deterministic steady state: $E[a] > a^*_{\text{det}}$ (precautionary saving)
- Measures aggregate buffer stock accumulation

Asset inequality $\sigma_a/E[a]$:

- Coefficient of variation of asset distribution
- Higher values indicate more asset inequality

Consumption smoothing σ_c/σ_y:

- Ratio of consumption volatility to income volatility
- Perfect smoothing: $\sigma_c/\sigma_y = 0$
- No smoothing: $\sigma_c/\sigma_y = 1$

Asset-income correlation $\text{Corr}(a, y)$:

- Positive: Rich tend to have high current income
- Close to 1: No consumption smoothing (hand-to-mouth)
- Close to 0: Perfect consumption smoothing

Summary: Steady State Solution

Key concepts:

1. **Stochastic steady state** $=$ time-invariant distribution, not fixed point
2. **Two-step solution**: First solve for policies, then find distribution
3. **Distribution iteration**: Track probability mass flows between states
4. **Economic insights**: Precautionary saving, asset inequality, partial smoothing

Taking stock

So far we have discussed how to find the

- optimal choices $g_c(a, y)$ and $g_a(a, y)$
- steady state distribution of agents $\mu^*(a, y)$

In the Huggett model a are risk free bonds and r is determined in equilibrium.

- Agents with $a < 0$ are borrowing from savers
- Agents with $a > 0$ are savers

Equilibrium: Bond market clears, such that total assets borrowed must equal total assets saved.

Market Clearing

Bond Market Clearing:

$$\int_{\mathcal{A}\times\mathcal{Y}} a\,\mu^*(da, dy) = 0$$

Interpretation:

- Aggregate asset holdings must equal zero
- For every saver, there must be a borrower
- But agents want to hold positive assets for precautionary reasons!

Labor Market: No labor market per se - productivity is exogenous

- Could interpret y as labor income endowment
- Aggregate labor supply $= \sum_j y_i \pi_j$ (constant)

Equilibrium Interest Rate

Define aggregate assets as

$$A(r) = \sum_{ij} a'(a_i, y_j; r)\, \mu^*(a_i, y_j; r)$$

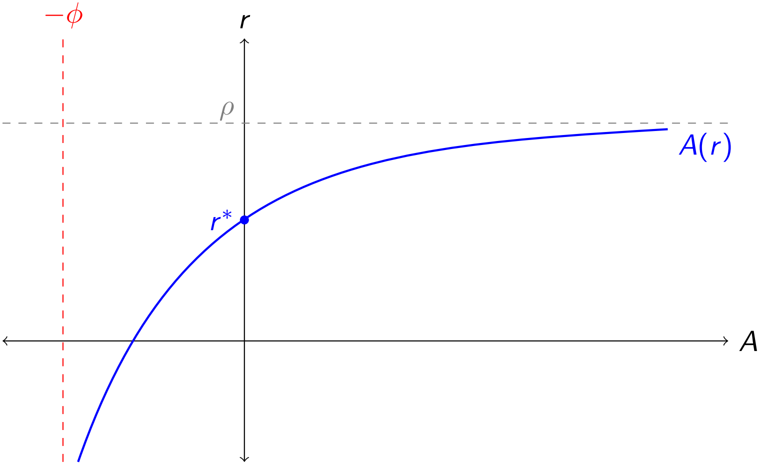
Market Clearing Determines r: The interest rate r^* must satisfy:

$$A(r^*) = \sum_{ij} a'(a_i, y_j; r^*)\, \mu^*(a_i, y_j; r^*) = 0$$

Key Insight:

- Agents want to save for precautionary reasons
- But in equilibrium, they cannot all be net savers
- Interest rate adjusts to clear the market
- Typically: $r^* < \rho$ where $\beta = \frac{1}{1+\rho}$

Aggregate Assets



Why $r^* < \rho$?

Complete Markets Benchmark: $r = \rho$ (no uncertainty about consumption)

Incomplete Markets:

- Agents face consumption risk
- Want to hold positive assets as insurance
- But assets must sum to zero in equilibrium
- Interest rate falls to discourage saving
- Lower r makes saving less attractive, borrowing more attractive

Intuition:

- "Precautionary saving premium" drives down interest rate
- Stronger precautionary motive $\Rightarrow$ lower equilibrium r

Comparative Statics

Increase in Risk (y more volatile):

- Stronger precautionary motive
- Interest rate falls
- More wealth inequality

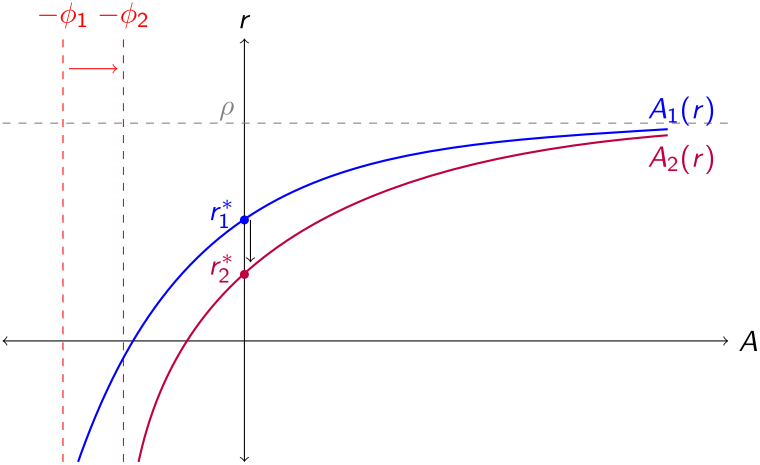
Increase in Persistence:

- Shocks last longer
- Greater need for self-insurance
- Interest rate falls, inequality rises

Tighter Borrowing Constraints:

- Less ability to smooth consumption
- Interest rate fall
- More agents at borrowing constraint

Tighter Borrowing Constraints $\phi_1 > \phi_2$



Computational Algorithm

Step 1: Discretize state space $\mathcal{A} \times \mathcal{Y}$

Step 2: For given r, solve individual problem

- Value function iteration on Bellman equation
- Obtain policy function $a'(a, y)$

Step 3: Find stationary distribution μ^*

- Iterate on distribution using policy function
- Continue until convergence

Step 4: Check market clearing

- Compute aggregate asset demand
- If > 0, increase r; if < 0, decrease r

Step 5: Repeat until market clears

Summary

Key Contributions:

- Framework for studying distributional macroeconomics
- Endogenous wealth inequality from incomplete markets
- New channel for interest rate determination

Mechanism:

- Idiosyncratic risk + incomplete markets $\Rightarrow$ precautionary saving
- Market clearing $\Rightarrow$ interest rate adjusts
- Heterogeneous outcomes despite identical preferences

Broader Impact:

- Foundation for modern heterogeneous agent macro
- Essential for distributional policy analysis
- Bridge between micro and macro economics