Consumption-Savings Under Uncertainty Cont. (Again)

Aiyagari: Adding Capital Accumulation

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Overview

Last Time:

- Huggett (1993)
- Heterogeneous agent model with borrowing
- Steady state distribution is stationary
- Aggregate assets determine the interest rate in equilibrium

Today:

- Aiyagari (1994)
- Production sector with firms demanding capital
- Interest rate determined in equilibrium, capital demanded = capital supplied

From Huggett to Aiyagari

Huggett (1993): Heterogeneous agents, incomplete markets, *no production*

- Exogenous interest rate or bond market clearing
- Focus on wealth distribution and precautionary saving

Aiyagari (1994): Adds **production sector** with capital accumulation

- Firms hire capital and labor competitively
- **Endogenous** factor prices (interest rate and wages)
- General equilibrium: household and firm optimization

Key Questions:

- How does incomplete insurance affect *aggregate* capital accumulation?
- What are the welfare costs of market incompleteness?
- How do distributional effects interact with production?

Main Result Preview

Central Finding: Economy **over-accumulates** capital relative to complete markets

Mechanism:

- Incomplete markets $\Rightarrow$ precautionary saving motive
- Agents want to hold positive assets for insurance
- In equilibrium: precautionary saving $=$ capital stock
- More capital than socially optimal $\Rightarrow$ $r < \rho$ and $MPK < \rho$

Welfare Implication:

- Too much saving crowds out consumption
- But incomplete insurance also costly
- Net welfare effect depends on parameters

Environment

Time: $t = 0, 1, 2, \ldots$ (discrete, infinite horizon)

Agents:

- Continuum of households of measure 1
- Continuum of firms of measure 1

Production Technology:

$$F(K, L) = K^{\alpha} L^{1-\alpha}$$

where $0 < \alpha < 1$, with constant returns to scale

Capital Depreciation: $\delta \in (0, 1)$ per period

Recource Constraint: $C + K' = F(K, L) + (1 - \delta)K$

Factor Prices (determined in equilibrium):

- Wage: $w = F_L(K, L) = (1 - \alpha)K^{\alpha} L^{-\alpha}$
- Rental rate: $r^k = F_K(K, L) - \delta = \alpha K^{\alpha-1} L^{1-\alpha} - \delta$

Household Problem

Preferences:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Individual State: (k, ε) where

- a: individual capital holdings (assets)
- y: idiosyncratic productivity shock

Productivity Process:

- $y \in \mathcal{Y} = \{y_1, y_2, , \ldots, y_N\}$
- Markov chain with transition matrix Π
- Stationary distribution π

Budget Constraint:

$$c + a' = wy + (1 + r)a$$

Borrowing Constraint: $a' \geq -\phi$

Interpretation of y

y represents productivity/efficiency, so:

- When agent has shock y, they supply y efficiency units of labor
- Higher $y \Rightarrow$ more productive worker $\Rightarrow$ higher effective labor supply
- This could represent: skill differences, health shocks, match quality with employer, etc.

Think of it as: Effective Labor$= y \times h$ where $h = 1$ (hours worked, normalized)

Aggregate labor constant (in steady state): $L = \mathbb{E}[y]$

Household's Bellman Equation

Taking aggregate capital K and labor L as given, the household solves:

$$V(a, y; K, L) = \max_{c,a'} \left\{ u(c) + \beta \sum_{y' \in \mathcal{Y}} \Pi(y, y') V(a', y'; K, L) \right\}$$

subject to:

$$\begin{aligned} c + a' &= w(K, L)y + (1 + r(K, L))a \\ a' &\geq -\phi \\ c &\geq 0 \end{aligned}$$

where $w(K, L)$ and $r(K, L)$ are equilibrium factor prices.

Policy Functions

Solution: Policy functions depend on aggregate state

$$g_a(a, y; K, L) : \text{capital choice}$$
$$g_c(a, y; K, L) : \text{consumption choice}$$

First Order Condition (when $a' > -\phi$):

$$u'(c) = \beta(1 + r(K, L)) \sum_{y' \in \mathcal{Y}} \Pi(y, y') u'(c')$$

Key Properties:

- $g_a(a, y; K, L)$ increasing in a
- $g_a(a, y; K, L)$ weakly increading in y
- Constraint $a' \geq -\phi$ may bind for low (a, y)

Aggregation

Distribution: $\mu(a, y)$ gives measure of agents with state (a, y)

Aggregate Capital:

$$K = \int a \, d\mu(a, y)$$

Aggregate Labor:

$$L = \int y \, d\mu(a, y) = \sum_{j=1}^{N} y_i \pi_j$$

Note: Aggregate labor is constant in steady state (depends only on π).

Law of Motion for Distribution:

$$\mu'(X, y') = \sum_{y \in \mathcal{Y}} \Pi(y, y') \int \mathbb{I}\{g_a(a, y; K, L) \in X\} \mu(da, y)$$

Market Clearing Conditions

1. **Capital Market Clearing**:

$$K' = \int g_a(a, y; K, L)\, d\mu(a, y)$$

Aggregate capital supply (household savings) = Aggregate capital demand (by firms)

2. **Labor Market Clearing**:

$$L = \int y\, d\mu(a, y)$$

Aggregate labor supply = Aggregate labor demand

3. **Goods Market Clearing**:

$$\int c(a, y; K, L)\, d\mu(a, y) + K' = F(K, L) + (1 - \delta)K$$

This is just the resource constraint.

Note: If two markets clear, the third clears automatically.

Factor Price Determination

Perfect Competition: Firms take factor prices as given and maximize profits

Firm's Problem:

$$\max_{K^d, L^d} F(K^d, L^d) - r^k K^d - wL^d$$

First Order Conditions:

$$r^k = F_K(K, L) - \delta = \alpha K^{\alpha-1} L^{1-\alpha} - \delta$$
$$w = F_L(K, L) = (1-\alpha) K^{\alpha} L^{-\alpha}$$

No-Arbitrage: $r = r^k$ (return on capital = interest rate)

Key Insight: Factor prices depend on aggregate quantities (K, L), which are determined by household decisions in equilibrium.

Definition: Stationary Recursive Competitive Equilibrium

A **Stationary Recursive Competitive Equilibrium** consists of:

1. **Value and Policy Functions**: $V(a, y)$, $g_a(a, y)$, $g_c(a, y)$
2. **Factor Prices**: r, w
3. **Aggregate Quantities**: K, L
4. **Stationary Distribution**: $\mu^*(a, y)$

such that:

Equilibrium Conditions

(i) Household Optimization: V, g_a, g_c solve the household's Bellman equation

(ii) Firm Optimization: Factor prices satisfy

$$r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta$$
$$w = (1-\alpha) K^{\alpha} L^{-\alpha}$$

(iii) Market Clearing: (in a stationary equilibrium $K = K'$)

$$K = \int g_a(a, y)\, d\mu^*(a, y)$$
$$L = \int y\, d\mu^*(a, y)$$

(iv) Consistency: μ^* is the stationary distribution implied by policy function $g_a(a, y)$

Equilibrium Characterization

Key Feature: Equilibrium (K, L, r, w) must be **self-consistent**

Fixed Point Problem:

- Given $(K, L) \Rightarrow$ compute (r, w) from firm FOCs
- Given $(r, w) \Rightarrow$ solve household problem for $g_a(a, y)$
- Given $g_a(a, y) \Rightarrow$ find stationary distribution μ^*
- Given $\mu^* \Rightarrow$ compute implied (K', L')
- Equilibrium: $(K', L') = (K, L)$

Existence and Uniqueness:

- Existence: Typically guaranteed under standard assumptions
- Uniqueness: Not guaranteed; multiple equilibria possible

The Equilibrium Condition

Equilibrium requires: $A(r) = K(r)$

Capital Demand (from firms): Always downward sloping

$$K(r) = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} L$$

Higher $r \to$ lower marginal product of capital needed $\to$ Firms demand less capital

Capital Supply (from households): May not be monotonic!

$$A(r) = \int g_a(a, y; r, w(r))\, d\mu^*(, y; r)$$

- Depends on household saving decisions
- Distribution μ^* is endogenous to r
- Complex interactions possible

How Does r Affect Household Saving?

When interest rate r increases, there are **three competing effects**:

1. **Substitution Effect** ($\uparrow r \Rightarrow \uparrow$ saving):
   - Higher return to saving makes future consumption cheaper
   - Standard price effect: save more
   - Increases A
2. **Income/Wealth Effect** ($\uparrow r \Rightarrow \downarrow$ saving):
   - Higher returns make *savers* wealthier
   - Savers increase their assets
   - Higher returns make *borrowers* poorer
   - Borrowers save less (borrow more)
   - Ambiguous effect on A, **depends on the distribution**

How Does r Affect Household Saving?

3. **Precautionary Motive** ($\uparrow r \Rightarrow \downarrow$ saving):
   - Higher r means buffer stock assets grow faster
   - Don't need as large a buffer for same insurance
   - Target wealth level falls
   - Decreases A

Intuition:

- At low r: need to hold many assets for precautionary reasons
- At high r: same insurance value with fewer assets

Net Effect is Ambiguous

Total Effect: $\frac{dA}{dr}$ = Substitution $\pm$ Income $-$ Precautionary

Possible Outcomes:

- **Substitution dominates and lots of savers**: A increasing in r
  - Common with high intertemporal elasticity of substitution (IES)
  - Yields unique equilibrium
- **Precautionary dominate or lots of borrowers**: A decreasing in r
  - Can occur with low IES, high risk aversion
  - Can still yield unique equilibrium (both curves downward)
- **Effects vary with r**: A non-monotonic
  - Different effects dominate at different interest rates
  - Can lead to multiple equilibria

Key Insight: Compared to the Huggett model, here we have a feedback loop through the production function that can lead to non-monotonicity.

Parameter Configurations Favoring Multiplicity

More Likely to Have Multiple Equilibria When:

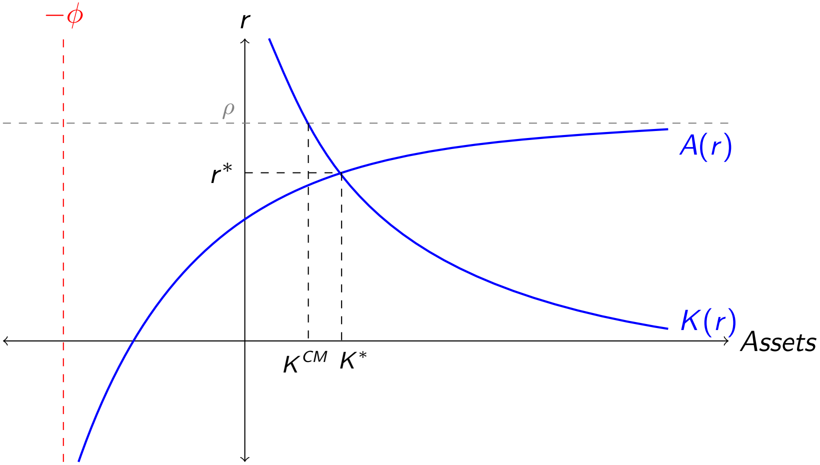
1. **High Risk Aversion** (γ large):
   - Strong income effects from interest rate changes
   - $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with large γ
2. **Very Persistent Income Shocks**:
   - Autocorrelation close to 1
   - Makes precautionary motive very sensitive to interest rates
3. **Loose Borrowing Constraints**:
   - Allows more heterogeneity in responses
   - ϕ large (can borrow significantly)
4. **Low Intertemporal Elasticity of Substitution (IES)**:
   - Income effects dominate substitution effects
   - IES $= \frac{1}{\gamma}$ for CRRA utility

Parameter Configurations Favoring Uniqueness

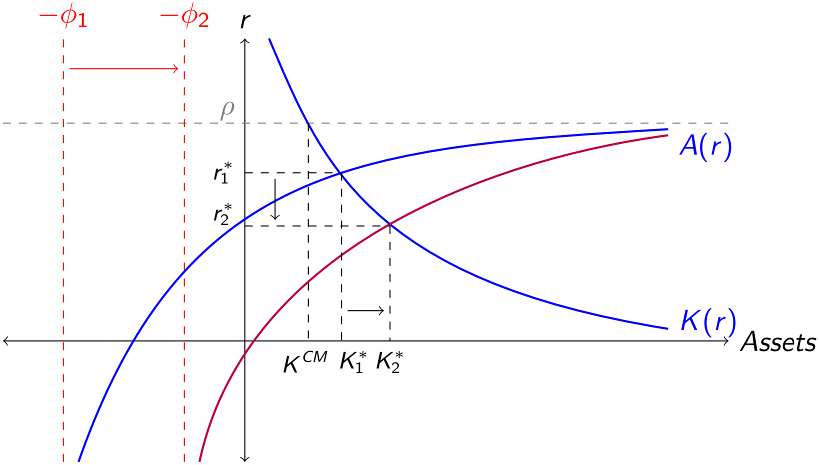
More Likely to Have Unique Equilibrium When:

1. **Log Utility** ($\gamma = 1$):
   - Income and substitution effects cancel exactly
   - IES $= 1$
   - Most robust case for uniqueness
2. **Low Persistence of Shocks**:
   - Weak precautionary motives
   - Less sensitivity to interest rates
3. **Tight Borrowing Constraints** ($\phi = 0$):
   - Forces similar saving behavior
   - Reduces heterogeneity in responses
4. **High IES** ($\gamma < 1$):
   - Substitution effects dominate
   - Clear positive relationship between r and A

Equilibrium (Let's assume $A(r)$ increasing in r)



Decrease in borrowing $\downarrow \phi$



The Challenge in Heterogeneous Agent Models

Individual vs. Aggregate State:

Individual State: (a, y)

- a: individual asset holdings
- y: idiosyncratic productivity shock
- What the household needs to know about itself

Aggregate State: ???

- What information about the aggregate economy matters?
- How do individual decisions depend on economy-wide variables?
- This is where the distribution comes in...

The Distribution $\mu(a, y)$:

- Describes the mass of agents at each state
- Evolves over time based on policy functions
- Is this a state variable we need to track?

Why the Distribution Matters

The distribution μ affects:

1. **Aggregate Capital**:
$$K = \int a \, d\mu(a, y)$$
2. **Aggregate Labor**:
$$L = \int y \, d\mu(a, y)$$
3. **Factor Prices**:
$$r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta$$
$$w = (1-\alpha) K^{\alpha} L^{-\alpha}$$

Conclusion: The distribution fundamentally affects individual decisions through factor prices

The Full State Space (Conceptually)

Theoretically, the complete state is:

Individual State: (k, ε, μ)

- a: own capital
- y: own productivity
- μ: distribution of all agents

Value Function:

$$V(a, y, \mu) = \max_{c, a'} \left\{ u(c) + \beta \sum_{y'} \Pi(y, y') V(a', y', \mu') \right\}$$

where $\mu' = T(\mu)$ is next period's distribution.

The Problem:

- μ is an **infinite-dimensional object** (a measure)
- Computing this is intractable!
- We need a way to avoid tracking μ explicitly

The Curse of Dimensionality

Why tracking μ is impossible:

Discretization Example:

- Suppose we discretize: $a \in \{a_1, \ldots, a_{100}\}$, $y \in \{y_1, y_2\}$
- The distribution μ has 200 dimensions (mass at each state)
- State space: $(a, y, \mu_1, \mu_2, \ldots, \mu_{200})$
- Value function has 202 arguments!

Computational Nightmare:

- Cannot store or interpolate in 200+ dimensions
- Would need astronomical memory
- Solution time would be prohibitive

We need a different approach

Key Insight: Steady State Assumption

The Trick: Focus on **stationary equilibria**

Stationary Equilibrium:

- Distribution is time-invariant: $\mu_t = \mu^*$ for all t
- Aggregate quantities constant: $K_t = K^*$, $L_t = L^*$
- Factor prices constant: $r_t = r^*$, $w_t = w^*$

Implication:

- μ is no longer a **dynamic state variable**
- It becomes an **endogenous outcome** of equilibrium
- We solve for μ^* as part of equilibrium, not as a state

Overview of Computational Approach

Challenge: Fixed point in *distribution space* - infinite dimensional object

Solution Strategy:

1. **Discretization**: Approximate continuous distributions with finite grids
2. **Nested Fixed Points**:
   - Outer loop: Find equilibrium (K, L)
   - Inner loop: Solve household problem and find stationary distribution
3. **Iteration**: Use fixed point iteration or other numerical methods

Key Steps:

- Discretize state spaces
- Solve household Bellman equation
- Compute stationary distribution
- Check market clearing
- Update aggregate quantities

Step 1: Discretization

Capital Grid: $\mathcal{A} = \{a_1, a_2, \ldots, a_{N_a}\}$

- Choose $a_1 = \phi$ (borrowing constraint)
- Choose a_{N_a} large enough to be non-binding
- Use non-uniform grids

Productivity Grid: $\mathcal{Y} = \{y_1, y_2, \ldots, y_{N_u}\}$

- Can use Tauchen (1986) method to discretize AR(1) process
- Or directly specify finite-state Markov chain

State Space: (a_i, y_j) for $i = 1, \ldots, N_a$ and $j = 1, \ldots, N_y$

Total States: $N_a \times N_y$ (typically 1000-5000 states)

Step 2: Solve Household Problem

Given: Aggregate state (K, L) and factor prices (r, w)

Value Function Iteration:

1. Initialize: $V^{(0)}(a_i, y_j) = 0$ for all (i, j)
2. For $n = 0, 1, 2, \ldots$ until convergence:

$$V^{(n+1)}(a_i, y_j) = \max_{k' \in \mathcal{K}} \Big\{ u(wy_j + (1+r)a_i - a') + \beta \sum_{\ell=1}^{N_y} \Pi_{j\ell} V^{(n)}(a', y_\ell) \Big\}$$

3. Store optimal policy: $g_a(a_i, y_j)$

Convergence Criterion: $\max_{i,j} |V^{(n+1)}(a_i, y_i) - V^{(n)}(a_i, y_j)| < \text{tol}$

Step 3: Find Stationary Distribution

Given: Policy function $g_a(a_i, y_j)$

Transition Matrix: Create $(N_k \times N_\varepsilon) \times (N_k \times N_\varepsilon)$ matrix Q
For state $(a_i, y_j) \to (a_\ell, y_m)$:

$$Q_{(i,j),(\ell,m)} = \begin{cases} \Pi_{jm} & \text{if } g_a(a_i, y_j) = k_\ell \\ 0 & \text{otherwise} \end{cases}$$

Stationary Distribution: Solve $\mu^* Q = \mu^*$ with $\sum \mu^* = 1$

- Iterate: $\mu^{(n+1)} = \mu^{(n)} Q$ until convergence

Alternative: faster methods exist for large state spaces

Step 4: Check Market Clearing

Compute Aggregate Quantities:

$$K' = \sum_{i=1}^{N_a} \sum_{j=1}^{N_y} g_a(a_i, y_j)\mu^*(a_i, y_j)$$

Market Clearing Errors:

$$\text{err}_K = |K' - K|$$

Convergence Check: If $\text{err}_K <$ tolerance, then STOP.

Otherwise, update K and repeat.

Step 5: Update Algorithm

Simple Updating:

$$K^{(n+1)} = \lambda K' + (1-\lambda)K^{(n)}$$

where $\lambda \in (0,1)$ is a damping parameter (typically 0.1-0.3)

Alternative Methods:

- **Bisection**: If only solving for K (since L is often fixed)
- **Newton-Raphson**: Compute derivatives numerically
- **Anderson Acceleration**: Faster convergence for smooth problems

Initial Guess:

- Start with complete markets capital stock: $K_0 = \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} L$
- Or use solution from simpler model (e.g., representative agent)

Complete Algorithm

Algorithm 1 Aiyagari Model Solution

Initialize: Grid $\mathcal{A}$, transition matrix Π, guess $(K^{(0)})$
Set $n = 0$
repeat
    Compute factor prices: $r^{(n)} = \alpha(K^{(n)})^{\alpha-1}(L)^{1-\alpha} - \delta$
    $w^{(n)} = (1-\alpha)(K^{(n)})^{\alpha}(L)^{-\alpha}$
    Solve household problem: VFI to get V, $g_a(a, y)$
    Find stationary distribution μ^*
    Compute implied aggregates:
    $K' = \sum_{i,j} g_a(a_i, y_j)\mu^*(a_i, y_j)$
    Check convergence: $|K' - K^{(n)}| < \text{tol}$
    Update: $K^{(n+1)} = \lambda K' + (1-\lambda)K^{(n)}$
    $n = n + 1$
until convergence
Return: Equilibrium (K^*, L^*, r^*, w^*), policy functions, distribution

What We Can and Cannot Analyze

Steady State Approach CAN Answer:

- Long-run wealth distribution
- Steady-state capital stock and interest rate
- Welfare in stationary equilibrium
- Comparative statics (how equilibrium changes with parameters)

Steady State Approach CANNOT Answer:

- Transitional dynamics after policy change
- Business cycle fluctuations
- Time-varying distributions
- Response to aggregate shocks

For dynamics, need different approaches:

- Perfect foresight transitions
- Krusell-Smith (1998) method
- Sequence space methods

Main Quantitative Results

Capital Over-Accumulation:

- Aiyagari finds $K^* > K^{CM}$ (complete markets benchmark)
- Over-accumulation of 10-40% depending on parameters
- $r^* < \rho$ due to precautionary saving

Interest Rate:

- Equilibrium interest rate below time preference rate
- $r^* = MPK - \delta < \rho$
- Gap depends on strength of precautionary motive

Wealth Distribution:

- Highly concentrated: top 20% hold 80-90% of wealth
- Many agents at borrowing constraint ($k = 0$)
- Realistic Gini coefficients (0.6-0.8)

Welfare Analysis

Competing Effects:

1. **Over-accumulation Cost**:
   - Too much capital $\Rightarrow$ too little consumption
   - Resources wasted on "excessive" investment
   - Golden rule: $MPK = \rho$ for optimal steady state

2. **Insurance Benefit**:
   - Higher capital stock $\Rightarrow$ higher wages
   - Partial self-insurance through asset accumulation
   - Reduces consumption volatility

Net Effect:

- Typically, over-accumulation cost dominates
- But welfare losses are small (1-2% of consumption)
- Depends on risk aversion, productivity variance, etc.

Modern Applications

1. **HANK Models**:
   - Heterogeneous Agent New Keynesian models
   - Kaplan, Moll, Violante (2018), others
   - Monetary policy transmission through wealth distribution
2. **Inequality and Growth**:
   - Endogenous skill formation and human capital
   - Entrepreneurship with borrowing constraints
   - Innovation and R&D with heterogeneous firms

Practical Implementation Tips

1. **Grid Construction**:
    - Use more grid points near borrowing constraint
    - Exponential spacing: $a_i = a_{\max} \left(\frac{i-1}{N_a - 1} \right)^{\nu}$ with $\nu > 1$
    - Check that maximum grid point is not binding in equilibrium
2. **Interpolation**:
    - Use linear interpolation for policy functions on off-grid points
    - Higher-order interpolation can cause oscillations
    - Monotonicity-preserving splines if needed
3. **Convergence**:
    - Use tight tolerance for VFI (10^{-6} or smaller)
    - Looser tolerance for outer loop (10^{-4})
    - Monitor convergence patterns - should be monotonic

Summary

Key Contributions of Aiyagari Model:

- Integrates heterogeneous agents with general equilibrium
- Shows how incomplete markets affect aggregate outcomes
- Provides framework for quantitative policy analysis

Main Insights:

- Precautionary saving leads to capital over-accumulation
- Incomplete insurance creates trade-offs for policy
- Distribution matters for aggregate quantities

Computational Legacy:

- Standard solution method for heterogeneous agent models
- Foundation for modern HANK models
- Continues to drive methodological innovations