Geometric Series and the Consumption Path

1 Solving the Geometric Series

When solving consumption-savings problems with CRRA utility, we often encounter sums that need to be simplified using geometric series formulas. Here are the key algebraic steps.

1.1 The Problem

From the budget constraint and optimal consumption growth, we derive: $c_0 \sum_{t=0}^{T-1} \left[\frac{\beta^{1/\sigma}}{(1+r)^{1-1/\sigma}}\right]^t = W_0$

The challenge is to solve this sum to find the initial consumption level c_0.

1.2 Step 1: Define the Common Ratio

Let $\phi = \frac{\beta^{1/\sigma}}{(1+r)^{1-1/\sigma}}$

Our equation becomes: $c_0 \sum_{t=0}^{T-1} \phi^t = W_0$

1.3 Step 2: Recognize the Geometric Series

The sum $\sum_{t=0}^{T-1} \phi^t$ is a finite geometric series with:

- First term: $\phi^0 = 1$
- Common ratio: ϕ
- Number of terms: T

1.4 Step 3: Apply the Geometric Series Formula

For any finite geometric series: $\sum_{t=0}^{T-1} \phi^t = \begin{cases} \frac{1-\phi^T}{1-\phi} & \text{if } \phi \neq 1 \\ T & \text{if } \phi = 1 \end{cases}$

Proof of this formula:

Let $S = 1 + \phi + \phi^2 + \cdots + \phi^{T-1}$

Multiply both sides by ϕ: $\phi S = \phi + \phi^2 + \phi^3 + \cdots + \phi^T$

Subtract the second equation from the first:

$$S - \phi S = (1 + \phi + \phi^2 + \cdots + \phi^{T-1}) - (\phi + \phi^2 + \phi^3 + \cdots + \phi^T) \quad (1)$$

$$(1 - \phi)S = 1 - \phi^T \quad (2)$$

$$S = \frac{1 - \phi^T}{1 - \phi} \quad (3)$$

1.5 Step 4: Solve for Initial Consumption

Substituting the geometric series formula: $c_0 \cdot \frac{1-\phi^T}{1-\phi} = W_0$

Therefore: $\boxed{c_0 = W_0 \cdot \frac{1-\phi}{1-\phi^T}}$

1.6 Economic Interpretation

The term $\frac{1-\phi}{1-\phi^T}$ is the **annuity factor** that converts total wealth into initial consumption:

- If $\phi < 1$: Consumption is front-loaded, so higher c_0
- If $\phi > 1$: Consumption is back-loaded, so lower c_0
- If $\phi = 1$: Flat consumption, so $c_0 = \frac{W_0}{T}$

1.7 Special Case: Flat Consumption

When $\phi = 1$ (which occurs when $\beta(1+r) = 1$): $\sum_{t=0}^{T-1} 1 = T$

So: $c_0 \cdot T = W_0$, giving $c_0 = \frac{W_0}{T}$

This makes economic sense: with flat consumption, each period receives an equal share of total wealth.

1.8 Connection to Present Value

Notice that $\phi = \frac{\beta^{1/\sigma}}{(1+r)^{1-1/\sigma}}$ can be rewritten as: $\phi = \left(\frac{\beta(1+r)}{(1+r)^\sigma}\right)^{1/\sigma} = \left(\frac{\beta}{(1+r)^{\sigma-1}}\right)^{1/\sigma}$

This shows how the geometric series solution connects to:

- **Discounting**: The β term represents time preference
- **Interest rate effects**: The $(1+r)$ terms capture intertemporal prices
- **Risk aversion**: The σ parameter determines sensitivity to these factors

The key insight is that optimal consumption growth creates a geometric pattern, allowing us to use closed-form series solutions rather than computing infinite sums.