Understanding Intertemporal Elasticity of Substitution: Infinite Substitution, Unit Elasticity, and No Substitution

1 Introduction

The intertemporal elasticity of substitution (IES) is a crucial parameter in macroeconomic models that measures how willing agents are to substitute consumption across time periods in response to changes in interest rates. For CRRA utility functions, the IES equals $\frac{1}{\sigma}$ where σ is the coefficient of relative risk aversion.

This document explains three important special cases:

1. **Infinite substitution** ($\sigma = 0$, IES $= \infty$)
2. **Unit elasticity** ($\sigma = 1$, IES $= 1$)
3. **No substitution** ($\sigma \to \infty$, IES $\to 0$)

2 Mathematical Framework

Consider the standard intertemporal consumption choice problem with CRRA utility:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

The Euler equation for optimal consumption is:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

With CRRA utility, this becomes:

$$c_t^{-\sigma} = \beta(1+r)c_{t+1}^{-\sigma}$$

Solving for consumption growth:

$$\frac{c_{t+1}}{c_t} = [\beta(1+r)]^{1/\sigma}$$

The intertemporal elasticity of substitution is defined as:

$$\text{IES} = \frac{d\ln(c_{t+1}/c_t)}{d\ln(1+r)} = \frac{1}{\sigma}$$

3 Case 1: Infinite Substitution ($\sigma = 0$)

3.1 Mathematical Properties

When $\sigma = 0$:

- Utility function: $u(c) = c$ (linear utility)
- Marginal utility: $u'(c) = 1$ (constant)
- Intertemporal elasticity: IES $= \frac{1}{0} = \infty$

The Euler equation becomes:

$$1 = \beta(1 + r) \cdot 1$$
$$\beta(1 + r) = 1$$

3.2 Economic Interpretation

Infinite substitution means agents are perfectly willing to rearrange consumption across time periods. Since there is no diminishing marginal utility:

- If $\beta(1 + r) > 1$ (even slightly): Agent consumes zero today, everything in the future
- If $\beta(1 + r) < 1$ (even slightly): Agent consumes everything today, zero in the future
- If $\beta(1 + r) = 1$ exactly: Agent is indifferent between any consumption profile

3.3 Why This Is Unrealistic

Linear utility leads to **corner solutions** that don't match real behavior:

Example: Suppose $\beta = 0.95$ and $r = 0.06$, so $\beta(1 + r) = 1.007$.

- The 0.7% advantage to waiting leads to consuming nothing today
- Real people would make small adjustments, not extreme ones
- This violates the consumption smoothing we observe empirically

3.4 Mathematical Intuition

With linear utility, the marginal rate of substitution between consumption today and tomorrow is:

$$MRS = \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{1}{\beta}$$

This is **independent of consumption levels**, so any deviation from the optimal interest rate leads to infinite adjustments.

4 Case 2: Unit Elasticity ($\sigma = 1$)

4.1 Mathematical Properties

When $\sigma = 1$:

- Utility function: $u(c) = \ln(c)$ (logarithmic utility)
- Marginal utility: $u'(c) = \frac{1}{c}$
- Intertemporal elasticity: $\text{IES} = 1$

The consumption growth formula becomes:

$$\frac{c_{t+1}}{c_t} = \beta(1 + r)$$

4.2 Economic Interpretation

Unit elasticity means a 1% change in the interest rate leads to a 1% change in consumption growth.

Examples:

- If $\beta = 0.95$ and $r = 5.26\%$: $\beta(1 + r) = 1$, so consumption is flat
- If interest rate rises to 15.26%: $\beta(1 + r) = 1.1$, consumption grows 10% per period
- If interest rate falls to -4.74%: $\beta(1 + r) = 0.9$, consumption falls 10% per period

4.3 Why This Is "Just Right"

Log utility provides a good balance:

- **Realistic responses**: Changes in interest rates lead to reasonable consumption adjustments
- **Interior solutions**: No corner solutions like linear utility
- **Mathematical convenience**: Log utility is analytically tractable
- **Empirical support**: IES estimates often cluster around 0.5-2, making 1 reasonable

5 Case 3: No Substitution ($\sigma \to \infty$)

5.1 Mathematical Properties

As $\sigma \to \infty$:

- Utility approaches $u(c) = \min\{c_0, c_1, c_2, \ldots\}$ (Leontief-like)
- Intertemporal elasticity: $\text{IES} \to 0$
- Consumption growth: $\frac{c_{t+1}}{c_t} = [\beta(1 + r)]^{1/\sigma} \to 1$

5.2 Economic Interpretation

No substitution means agents keep consumption perfectly flat regardless of interest rates.

The agent has an **extreme preference for consumption smoothing**:

- Any deviation from flat consumption creates large utility losses
- Interest rate changes don't affect consumption timing decisions
- Agent acts as if they have "consumption habits" or rigid lifestyle preferences

5.3 Behavioral Examples

Think of agents who:

- Want exactly the same standard of living every period
- Are unwilling to sacrifice current consumption even for high returns
- View consumption variability as extremely costly
- Have strong "consumption habits" or social pressures for stable lifestyles

Extreme Example: Even if the interest rate is 50%, the agent still maintains flat consumption because the "pain" of lower consumption today outweighs the benefits of much higher consumption tomorrow.

5.4 When This Might Be Realistic

No substitution could apply to:

- **Subsistence-level consumption**: Basic needs cannot be postponed
- **Habit formation**: Strong preferences for consistent lifestyles
- **Social constraints**: Pressure to maintain social status
- **Liquidity constraints**: Unable to borrow against future income

6 Comparative Analysis

Property	Infinite ($\sigma = 0$)	Unit ($\sigma = 1$)	None ($\sigma \to \infty$)
IES	∞	1	0
Utility	c	$\ln(c)$	$\min\{c_t\}$
Interest response	Extreme	Moderate	None
Consumption path	Corner solutions	Smooth adjustment	Always flat
Realism	Unrealistic	Reasonable	Sometimes realistic

Table 1: Comparison of Substitution Cases

7 Conclusion

The intertemporal elasticity of substitution is a fundamental parameter that determines how agents respond to changes in interest rates and intertemporal prices. Understanding the three limiting cases helps build intuition:

1. **Infinite substitution** ($\sigma = 0$): Leads to unrealistic corner solutions but shows the importance of diminishing marginal utility

2. **Unit elasticity** ($\sigma = 1$): Provides a reasonable baseline with moderate responses to interest rate changes

3. **No substitution** ($\sigma \to \infty$): Shows the extreme case where consumption smoothing dominates all other considerations

In practice, most macroeconomic models use intermediate values of σ (typically 1-5) that provide realistic behavior while maintaining analytical tractability. The choice of σ has important implications for policy analysis and should be carefully considered based on the specific context and empirical evidence.