EC9A2 Problem Set 1 Solutions

Fall 2025

Instructions

- Show all work clearly and provide economic intuition for your results.
- For analytical problems, derive solutions step by step.
- For the MATLAB problem, submit both your code and a brief write-up of results.

1 Finite Horizon Consumption-Savings

Consider a consumer who lives for $T = 3$ periods with initial wealth $W_0 = 100$ and no income. The consumer has CRRA utility $u(c) = \frac{c^{1-\theta}}{1-\theta}$ with $\theta = 2$, discount factor $\beta = 0.9$, and faces interest rate $r = 0.05$.

(a) Using the Lagrangian method, derive the present value budget constraint and set up the optimization problem.

ANSWER: Starting with the budget constraint $W_{t+1} = (1+r)(W_t - c_t)$, we iterate forward:

$$W_1 = (1+r)(W_0 - c_0) \tag{1}$$
$$W_2 = (1+r)(W_1 - c_1) = (1+r)^2(W_0 - c_0) - (1+r)c_1 \tag{2}$$
$$W_3 = (1+r)(W_2 - c_2) = (1+r)^3(W_0 - c_0) - (1+r)^2c_1 - (1+r)c_2 \tag{3}$$

With terminal condition $W_3 \geq 0$ (equality at optimum):

$$(1+r)^3c_0 + (1+r)^2c_1 + (1+r)c_2 = (1+r)^3W_0 \tag{4}$$

Dividing by $(1+r)^3$:

$$c_0 + \frac{c_1}{(1+r)} + \frac{c_2}{(1+r)^2} = W_0 = 100 \tag{5}$$

Optimization Problem:

$$\max_{c_0,c_1,c_2} \sum_{t=0}^{2} \beta^t \frac{c_t^{1-\theta}}{1-\theta} \quad \text{subject to} \quad \sum_{t=0}^{2} \frac{c_t}{(1+r)^t} = W_0 \tag{6}$$

(b) Derive the Euler equation and solve for the consumption growth rate $\frac{c_{t+1}}{c_t}$.

ANSWER: Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{2} \beta^t \frac{c_t^{1-\theta}}{1-\theta} - \lambda \left(\sum_{t=0}^{2} \frac{c_t}{(1+r)^t} - W_0 \right) \tag{7}$$

First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t c_t^{-\theta} - \frac{\lambda}{(1+r)^t} = 0 \tag{8}$$

Therefore: $\beta^t c_t^{-\theta} = \frac{\lambda}{(1+r)^t}$
Deriving Euler equation:

$$\beta^t c_t^{-\theta}(1+r)^t = \lambda \tag{9}$$

$$\beta^{t+1} c_{t+1}^{-\theta}(1+r)^{t+1} = \lambda \tag{10}$$

Equating and simplifying:

$$c_t^{-\theta} = \beta(1+r)c_{t+1}^{-\theta} \tag{11}$$

Consumption growth rate:

$$\frac{c_{t+1}}{c_t} = [\beta(1+r)]^{1/\theta} \tag{12}$$

With $\theta = 2$, $\beta = 0.9$, $r = 0.05$:

$$\frac{c_{t+1}}{c_t} = [0.9 \times 1.05]^{1/2} = [0.945]^{0.5} = 0.972 \tag{13}$$

Economic interpretation: Consumption declines over time since $\beta(1+r) < 1$, meaning impatience dominates the interest rate.

(c) Calculate the consumption level in each period (c_0, c_1, c_2) and verify that the budget constraint is satisfied.

ANSWER: With geometric consumption growth: $c_t = c_0 \cdot (0.972)^t$. Substituting into the budget constraint:

$$\frac{c_0}{1.05} + \frac{c_0 \cdot 0.972}{(1.05)^2} + \frac{c_0 \cdot (0.972)^2}{(1.05)^3} = 100 \tag{14}$$

$$c_0 \left[\frac{1}{1.05} + \frac{0.972}{1.1025} + \frac{0.945}{1.158} \right] = 100 \tag{15}$$

$$c_0[0.952 + 0.882 + 0.816] = 100 \tag{16}$$

$$c_0 \cdot 2.650 = 100 \Rightarrow c_0 = 37.74 \tag{17}$$

Consumption path:

$$c_0 = 37.74 \tag{18}$$

$$c_1 = 37.74 \times 0.972 = 36.68 \tag{19}$$

$$c_2 = 36.68 \times 0.972 = 35.65 \tag{20}$$

Verification: Present value $= \frac{37.74}{1.05} + \frac{36.68}{1.1025} + \frac{35.65}{1.158} = 35.94 + 33.27 + 30.79 = 100.00$

2 Infinite Horizon Analysis

Consider an infinite horizon consumption-savings problem with the following setup:

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c_t) \tag{21}$$

subject to:

$$a_{t+1} = (1+r)a_t + y - c_t \tag{22}$$
$$c_t \geq 0 \tag{23}$$
$$a_t \geq \underline{a} \tag{24}$$
$$\lim_{t\to\infty} \beta^t u'(c_t)a_t = 0 \tag{25}$$

(a) Explain the economic meaning of each constraint.

ANSWER: Constraints explanation:

- $a_{t+1} = (1+r)a_t + y - c_t$: Standard budget constraint where assets evolve based on interest earnings, income, and consumption.
- $c_t \geq 0$: Non-negativity constraint on consumption (survival constraint).
- $a_t \geq 0$: This is the borrowing constraint. In this case agents cannot borrow assets.
- Transversality condition: Prevents explosive debt or asset accumulation paths that would be inconsistent with optimization.

(b) Write down the Bellman equation and derive the first-order condition envelope condition, and Euler equation.

ANSWER: Bellman equation:

$$V(a) = \max_{c\geq 0} \{\ln(c) + \beta V((1+r)a + y - c)\} \tag{26}$$

subject to: $(1+r)a + y - c \geq 0$
First-order condition:

$$\frac{\partial}{\partial c}[\ln(c) + \beta V((1+r)a + y - c)] = 0 \tag{27}$$

$$\frac{1}{c} - \beta V'((1+r)a + y - c) = 0 \tag{28}$$

$$\frac{1}{c} = \beta V'(a') \tag{29}$$

Envelope condition:

$$V'(a) = \frac{\partial}{\partial a}[\ln(c^*(a)) + \beta V((1+r)a + y - c^*(a))] \tag{30}$$

Since $c^*(a)$ is optimal, by the envelope theorem:

$$V'(a) = \beta V'\left((1+r)a + y - c^*(a)\right) \cdot (1+r) \tag{31}$$

$$V'(a) = \beta(1+r)V'(a') \tag{32}$$

Deriving the Euler equation: From the first-order condition at time t: $\frac{1}{c_t} = \beta V'(a_{t+1})$
From the first-order condition at time $t+1$: $\frac{1}{c_{t+1}} = \beta V'(a_{t+2})$
From the envelope condition: $V'(a_{t+1}) = \beta(1+r)V'(a_{t+2})$
Substituting the envelope condition into the time t FOC:

$$\frac{1}{c_t} = \beta \cdot \beta(1+r)V'(a_{t+2}) = \beta^2(1+r)V'(a_{t+2}) \tag{33}$$

But from the time $t+1$ FOC: $V'(a_{t+2}) = \frac{1}{\beta c_{t+1}}$
Therefore:

$$\frac{1}{c_t} = \beta^2(1+r) \cdot \frac{1}{\beta c_{t+1}} = \frac{\beta(1+r)}{c_{t+1}} \tag{34}$$

Final Euler equation:

$$\frac{1}{c_t} = \frac{\beta(1+r)}{c_{t+1}} \tag{35}$$

(c) Suppose $\beta(1+r) = 1$. Show that the transversality condition becomes $\lim_{t\to\infty}\left(\frac{1}{1+r}\right)^t a_t = 0$. What does this imply about the agent's asset holding behavior in the long run?

ANSWER: When $\beta(1+r) = 1$, we have from the Euler equation:

$$\frac{1}{c_t} = \beta(1+r)\frac{1}{c_{t+1}} = \frac{1}{c_{t+1}}$$

.

This implies $c_t = c_{t+1}$ for all t (constant consumption). Transversality condition:

$$\lim_{t\to\infty} \beta^t u'(c_t) a_t = 0$$

With constant consumption and $u'(c) = \frac{1}{c}$:

$$\lim_{t\to\infty} \beta^t \frac{1}{c} a_t = \frac{1}{c} \lim_{t\to\infty} \beta^t a_t = 0 \tag{36}$$

Since $c > 0$, this requires: $\lim_{t\to\infty} \beta^t a_t = 0$. With $\beta(1+r) = 1$, we have $\beta = \frac{1}{1+r}$, so:

$$\lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t a_t = 0 \tag{37}$$

This means assets cannot grow faster than rate r. In the long run, the agent holds just enough assets to maintain constant consumption from the income stream.

(d) Now suppose $\beta(1 + r) > 1$. Analyze what happens to the consumption path over time. Is this economically reasonable? Explain why the transversality condition is necessary to rule out explosive behavior.

ANSWER: When $\beta(1 + r) > 1$, the Euler equation gives:

$$\frac{c_{t+1}}{c_t} = \beta(1 + r) > 1 \tag{38}$$

Consumption path: Consumption grows without bound, which is unrealistic for several reasons:

1. Resource constraints: Infinite consumption growth requires infinite wealth accumulation.
2. Diminishing marginal utility: As consumption gets very large, additional units provide little extra satisfaction.
3. Market equilibrium: If all agents tried to consume infinitely, markets would not clear.

Role of transversality condition: Without the transversality condition, the agent could achieve infinite utility by doing the following:

1. Letting wealth compound faster than the discount rate
2. Consuming the growing wealth stream

The transversality condition rules this out by requiring that either assets don't grow too fast (if $a_t > 0$). The key insight is that the transversality condition constrains the growth rate of assets. This ensures the solution has finite consumption and is economically meaningful.

3 MATLAB Implementation

Implement value function iteration to solve the infinite horizon consumption-savings problem numerically using the following parameters:

- Utility: $u(c) = \ln(c)$
- $\beta = 0.9$, $r = 0.05$, $y = 1$
- Asset grid: 500 points from $a_{min} = 0$ to $a_{max} = 50$
- Convergence tolerance: 10^{-6}

(a) Write MATLAB code that implements the value function iteration algorithm. Your code should:

- Set up the asset grid and initial value function guess
- Implement the Bellman operator
- Iterate until convergence
- Store both the value function and policy functions

ANSWER: See solution MATLAB code.

(b) Create plots showing:

- The converged value function $V(a)$
- The optimal policy functions $g_c(a)$ and $g_a(a)$