EC9A2 Problem Set 2

1 Comparative Statics with Depreciation Rate (δ)

Consider the Ramsey model with exogenous labor supply (normalized to 1). The economy is characterized by:

- Preferences: $U = \sum_{t=0}^{\infty} \beta^t u(c_t)$ where $\beta \in (0, 1)$
- Technology: $y_t = f(k_t)$ with $f'(k) > 0$, $f''(k) < 0$
- Capital accumulation: $k_{t+1} = (1 - \delta)k_t + i_t$
- Resource constraint: $c_t + i_t = f(k_t)$

(a) Derive the steady-state conditions for capital (k^*) and consumption (c^*) as functions of the model parameters. (You can start from the first oder conditions, you do not need to derive them.)

ANSWER: In steady state, all variables are constant: $k_{t+1} = k_t = k^*$ and $c_{t+1} = c_t = c^*$. From the household's Euler equation and the firm optimization ($r_{t+1} = f'(k_{t+1})$:

$$u'(c_t) = \beta u'(c_{t+1})[1 - \delta + f'(k_{t+1})]$$

In steady state, $c_t = c_{t+1} = c^*$, so:

$$u'(c^*) = \beta u'(c^*)[1 - \delta + f'(k^*)]$$

Since $u'(c^*) > 0$, we can divide both sides:

$$1 = \beta[1 - \delta + f'(k^*)]$$

Solving for the marginal product of capital:

$$f'(k^*) = \frac{1}{\beta} - 1 + \delta = \frac{1 - \beta}{\beta} + \delta = \rho + \delta$$

where $\rho = \frac{1-\beta}{\beta}$ is the subjective discount rate. Combining the resource constraint $c_t + i_t = f(k_t)$ with the capital accumulation equation $k_{t+1} = (1 - \delta)k_t + i_t$:

$$k_{t+1} = (1 - \delta)k_t + f(k_t) - c_t \tag{1}$$

In steady state, $k_{t+1} = k_t = k^*$:

$$k^* = (1 - \delta)k^* + f(k^*) - c^* \tag{2}$$

Solving for steady-state consumption:

$$c^* = f(k^*) - \delta k^* \tag{3}$$

Summary: The steady state is characterized by:

$$\begin{aligned} f'(k^*) &= \rho + \delta && \text{(determines } k^*\text{)} \\ c^* &= f(k^*) - \delta k^* && \text{(determines } c^* \text{ given } k^*\text{)} \end{aligned}$$

(b) Analyze the effect of an increase in the depreciation rate δ on the steady state using phase diagram analysis. Show algebraically and graphically how the $\Delta k = 0$ locus and $\Delta c = 0$ change when δ increases from δ_L to δ_H. Illustrate the new steady state (k^{**}, c^{**}) compared to the original (k^*, c^*) on your phase diagram.

ANSWER: The $\Delta k = 0$ locus is defined by $k_{t+1} = k_t$, which gives:

$$c = f(k) - \delta k \tag{4}$$

Taking the derivative with respect to δ:

$$\left.\frac{\partial c}{\partial \delta}\right|_{\Delta k = 0} = -k < 0$$

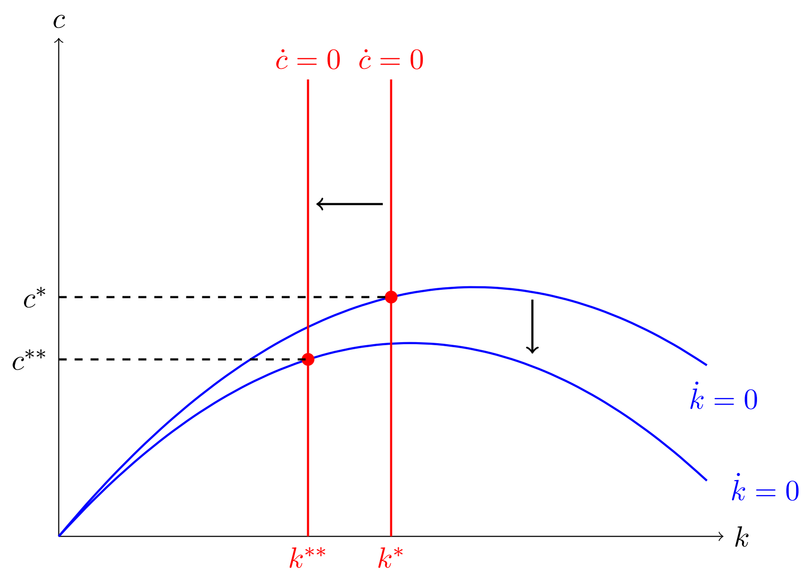
For any given k, an increase in δ from δ_L to δ_H decreases the consumption. The locus shifts **downward** by exactly $k \cdot (\delta_H - \delta_L)$ at each point. The peak occurs where $f'(k) = \delta$ so when δ increases, the peak moves to the left and downward.

The $\Delta c = 0$ locus is defined by $c_{t+1} = c_t$. From the Euler equation in steady state: $f'(k^*) = \rho + \delta$. This defines a vertical line at $k = k^*$ where $f'(k^*) = \rho + \delta$. When δ increases from δ_L to δ_H:

$$f'(k^{**}) = \rho + \delta_H > \rho + \delta_L = f'(k^*)$$

By the concavity of f ($f'' < 0$), this requires $k^{**} < k^*$. So the $\Delta c = 0$ locus shifts **leftward** (to a lower capital stock).

The new steady state (k^{**}, c^{**}) is at the intersection of the new loci and both capital and consumption fall.



(c) Assuming the economy is initially in steady state at (k^*, c^*) when δ increases unexpectedly and permanently to δ_H, describe and illustrate the transition dynamics. What happens to capital and consumption on impact? How does the economy move to the new steady state?

ANSWER: Assuming the economy is initially at (k^*, c^*) when δ increases unexpectedly. $k_0 = k^*$ since capital is predetermined and cannot jump. c_0 jumps immediately to place the economy on the new saddle path. Capital decumulates: k_t falls gradually from k^* toward k^{**} and consumption slowly falls to its new steady state level c^{**}.

(d) Why does higher depreciation affect the steady-state capital stock the way it does? Explain the mechanism through the Modified Golden Rule condition.

ANSWER: The Modified Golden Rule condition is:

$$f'(k^*) = \rho + \delta \tag{5}$$

This equation states that at the optimal steady state, the marginal product of capital equals the "required return," which is the sum of impatience (ρ) and depreciation (δ).

When δ increases:

- The required return $\rho + \delta$ increases
- To maintain the equality, $f'(k^*)$ must increase
- Since $f'' < 0$ (diminishing marginal product), this requires k^* to decrease

Economic mechanism: Higher depreciation means that each unit of capital produces less net return after accounting for depreciation losses. To make capital accumulation worthwhile, agents need a higher gross marginal product, which occurs at lower capital levels.

2 Equivalence of Planner's Problem and Competitive Equilibrium

Consider the Ramsey model with endogenous labor supply. The economy has:

- Preferences: $U = \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - \ell_t)$
- Technology: $Y_t = F(K_t, L_t)$ with constant returns to scale
- Capital accumulation: $K_{t+1} = (1 - \delta)K_t + I_t$
- Time endowment: $\ell_t \in [0, 1]$

(a) Given technology and the resource constraint, the social planner wants to maximize total utility. Write down the social planner's problem in discrete time. State the objective function the planner maximizes and write all the constraints. Clearly identify which variables are choice variables and which are state variables.

ANSWER: The social planner maximizes the present discounted value of utility:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - \ell_t)$$

where $\beta \in (0, 1)$ is the discount factor, c_t is consumption, and ℓ_t is labor supply (with $1 - \ell_t$ being leisure) at time t.

The resource constraint:

$$c_t + I_t = F(K_t, L_t) \quad \forall t$$

Capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + I_t \quad \forall t$$

Combining these two constraints (substituting out I_t):

$$K_{t+1} = (1 - \delta)K_t + F(K_t, L_t) - c_t \quad \forall t$$

This is the key constraint that links periods together.

Choice (control) variables: $\{c_t, L_t, K_{t+1}\}_{t=0}^{\infty}$ Alternatively, since K_{t+1} is determined by the resource constraint once we choose c_t and L_t, we could treat $\{c_t, L_t\}_{t=0}^{\infty}$ as the choice variables with K_{t+1} being determined residually. However, for the Lagrangian approach, it's cleaner to treat all three as choice variables with the resource constraint as an explicit constraint.

State variable: K_t (capital stock at the beginning of period t)

Initial condition: K_0 is given

Transversality condition:

$$\lim_{t\to\infty} \beta^t u_c(c_t, 1 - L_t)K_{t+1} = 0$$

This condition ensures that the planner doesn't accumulate infinite capital (which would be wasteful) or die with positive debt (which would be infeasible). It states that the present value of terminal capital must go to zero.

The planner's problem can be written as:

$$\begin{aligned}
&\max_{\{c_t, L_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - L_t) \\
&\text{subject to: } K_{t+1} = (1-\delta)K_t + F(K_t, L_t) - c_t \quad \forall t \\
&\qquad K_0 \text{ given} \\
&\qquad \text{transversality condition}
\end{aligned}$$

(b) Solve the social planner's problem using the Lagrangian method. Derive the first-order conditions and solve for the planners Euler equation and intratemporal optimality condition.

ANSWER: The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \{u(c_t, 1 - L_t) + \lambda_t[(1-\delta)K_t + F(K_t, L_t) - c_t - K_{t+1}]\}$$

where λ_t is the Lagrange multiplier on the resource constraint at time t. Note that λ_t represents the shadow value of relaxing the resource constraint (or equivalently, the marginal utility value of an additional unit of the resource in period t).

FOC with respect to c_t:

Taking the derivative of $\mathcal{L}$ with respect to c_t:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u_c(c_t, 1 - L_t) - \beta^t \lambda_t = 0$$

Therefore:

$$u_c(c_t, 1 - L_t) = \lambda_t$$

FOC with respect to L_t:

Taking the derivative with respect to L_t:

$$\frac{\partial \mathcal{L}}{\partial L_t} = -\beta^t u_{1-L}(c_t, 1 - L_t) + \beta^t \lambda_t F_L(K_t, L_t) = 0$$

Therefore:

$$u_{1-L}(c_t, 1 - L_t) = \lambda_t F_L(K_t, L_t)$$

FOC with respect to K_{t+1}:

Taking the derivative with respect to K_{t+1}:

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1}[(1-\delta) + F_K(K_{t+1}, L_{t+1})] = 0$$

Therefore:

$$\lambda_t = \beta\lambda_{t+1}[F_K(K_{t+1}, L_{t+1}) + 1 - \delta]$$

Planner's Euler Equation:
From the FOC with respect to c_t: $u_c(c_t, 1 - L_t) = \lambda_t$
Similarly for period $t + 1$: $u_c(c_{t+1}, 1 - L_{t+1}) = \lambda_{t+1}$
Substituting these into the FOC with respect to K_{t+1} we get the Euler equation:

$$u_c(c_t, 1 - L_t) = \beta u_c(c_{t+1}, 1 - L_{t+1})[F_K(K_{t+1}, L_{t+1}) + 1 - \delta]$$

Planner's Intratemporal Condition:
Dividing the FOC with respect to L_t by the FOC with respect to c_t:

$$\frac{u_{1-L}(c_t, 1 - L_t)}{u_c(c_t, 1 - L_t)} = F_L(K_t, L_t)$$

(c) Recall that the for the decentralized competitive equilibrium what we solved in lecture, the household optimality conditions are:

$$u_c(c_t, 1 - \ell_t) = \beta u_c(c_{t+1}, 1 - \ell_{t+1})[r_{t+1} + 1 - \delta]$$

$$\frac{u_{1-\ell}(c_t, 1 - \ell_t)}{u_c(c_t, 1 - \ell_t)} = w_t$$

When are these the same as the social planner's optimality conditions?

ANSWER: The social planner's optimality conditions are equal to the decentralized optimality conditions when

$$r_{t+1} = F_K(K_{t+1}, L_{t+1})$$

$$w_t = F_L(K_t, L_t)$$

That is, when prices are marginal products.

(d) Show that when $F(K, L)$ exhibits constant returns to scale, firm profits are zero in equilibrium, and verify that the resource constraints are equivalent in both problems.

ANSWER: With constant returns to scale (CRS), the production function satisfies:

$$F(\lambda K, \lambda L) = \lambda F(K, L) \quad \forall \lambda > 0$$

A function that constant returns to scale satisfies:

$$F(K, L) = F_K(K, L) \cdot K + F_L(K, L) \cdot L$$

Proof: Start with the CRS property: $F(\lambda K, \lambda L) = \lambda F(K, L)$
Differentiate both sides with respect to λ:

$$\frac{\partial F(\lambda K, \lambda L)}{\partial \lambda} = F(K, L)$$

Using the chain rule on the left side:

$$F_K(\lambda K, \lambda L) \cdot K + F_L(\lambda K, \lambda L) \cdot L = F(K, L)$$

Setting $\lambda = 1$ gives us a property about our original production function:

$$F_K(K, L) \cdot K + F_L(K, L) \cdot L = F(K, L)$$

Firm profits in equilibrium:
In equilibrium, markets clear so $K_t^d = K_t$ and $L_t^d = L_t$. Using the firm's FOCs and our property from above:

$$\begin{aligned}
\pi_t &= F(K_t, L_t) - r_t K_t - w_t L_t \\
&= F(K_t, L_t) - F_K(K_t, L_t) K_t - F_L(K_t, L_t) L_t \\
&= F(K_t, L_t) - F(K_t, L_t) \\
&= 0
\end{aligned}$$

Therefore, **profits are zero in equilibrium** with constant returns to scale. This is a fundamental result: under perfect competition and CRS, firms earn zero economic profits in equilibrium because all output is exhausted by factor payments.

The household's budget constraint is:

$$c_t + k_{t+1} = r_t k_t + w_t \ell_t + (1 - \delta) k_t$$

In equilibrium with market clearing ($k_t = K_t$ and $\ell_t = L_t$) and competitive prices ($r_t = F_K(K_t, L_t)$ and $w_t = F_L(K_t, L_t)$):

$$c_t + K_{t+1} = F_K(K_t, L_t) K_t + F_L(K_t, L_t) L_t + (1 - \delta) K_t$$

Using the property of CRS function:

$$c_t + K_{t+1} = F(K_t, L_t) + (1 - \delta) K_t$$

Rearranging:

$$K_{t+1} = (1 - \delta) K_t + F(K_t, L_t) - c_t$$

This is **identical to the planner's resource constraint**.

(e) Explain why constant returns to scale is crucial for the equivalence between the social planner's problem and the decentralized problem. What role does zero profit play?

ANSWER: Constant returns to scale is crucial for equivalence because:

1. Zero profits ensure correct factor payments:

- With CRS, Euler's theorem guarantees: $F(K, L) = F_K \cdot K + F_L \cdot L$
- All output is paid to factors: $Y = rK + wL$
- Households receive the full value of their contributions
- No residual profits or losses distort incentives

2. Private and social returns to factors coincide:

- Competitive factor prices equal marginal products: $r = F_K$ and $w = F_L$
- The private return households receive equals the social marginal product
- No externalities: private decisions internalize all social costs and benefits

3. Budget constraint equals resource constraint:

What would fail without CRS:

If the production function had increasing or decreasing returns to scale, and we'd have:

$$F(K, L) \neq F_K \cdot K + F_L \cdot L$$

This creates a wedge between private incentives and social optimality. For the social optimum and decentralized optimum to equal in this case, households must also receive profits from the firms, along with factor payments.