EC9A2 Problem Set 4

Model Setup

Consider an infinitely-lived representative household that maximizes expected lifetime utility:

$$\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$

where the period utility function is:

$$u(c, h) = \frac{c^{1-\sigma}}{1-\sigma} - \chi \cdot h$$

The household faces the budget constraint:

$$c_t + k_{t+1} = f(k_t, h_t) + (1-\delta)k_t$$

The production function is:

$$f(k, h) = k^\alpha h^{1-\alpha}$$

Key features:

- $h_t \in \{0, \bar{h}\}$ is the discrete labor choice (not work or work full-time)
- $\bar{h} = 0.4$ is the time endowment devoted to work when working
- $\chi = 0.5$ is the disutility of working (captures commuting costs, loss of leisure, etc.)
- $\beta = 0.96$
- $\sigma = 2$
- $\alpha = 0.33$
- $\delta = 0.1$

1 Theoretical Analysis

(a) Write the household's problem in recursive form using a value function $V(k)$. Be explicit about the discrete choice over $h \in \{0, \bar{h}\}$.

ANSWER: The household's problem in recursive form is:

$$V(k) = \max_{h \in \{0,\bar{h}\}, k' \geq 0} \{u(c, h) + \beta V(k')\}$$

subject to:

$$c = f(k, h) + (1 - \delta)k - k'$$

(b) Show that the value function can be written as:

$$V(k) = \max\{V^W(k), V^N(k)\}$$

where $V^W(k)$ is the value of working and $V^N(k)$ is the value of not working. Write out the Bellman equations for both $V^W(k)$ and $V^N(k)$ explicitly.

ANSWER: We can rewrite the value function as:

$$V(k) = \max\left\{ \max_{k' \geq 0} \left\{u(c, \bar{h}) + \beta V(k')\right\}, \max_{k' \geq 0} \left\{u(c, 0) + \beta V(k')\right\} \right\}$$
$$V(k) = \max\{V^W(k), V^N(k)\}$$

Where the value function for working ($h = \bar{h}$) is:

$$V^W(k) = \max_{k' \geq 0} \left\{ \frac{c^{1-\sigma}}{1-\sigma} - \chi\bar{h} + \beta V(k') \right\}$$

subject to:

$$c = k^\alpha \bar{h}^{1-\alpha} + (1 - \delta)k - k'$$

And the value function for not working ($h = 0$) is:

$$V^N(k) = \max_{k' \geq 0} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta V(k') \right\}$$

subject to:

$$c = (1 - \delta)k - k'$$

Note that when $h = 0$, there is no production output, only depreciated capital.

(c) For each labor choice $h \in \{0, \bar{h}\}$, derive the Euler equation relating optimal capital accumulation today to next period's capital. Use the envelope theorem to express $\frac{\partial V(k)}{\partial k}$ for each labor choice.

ANSWER: For the working problem, the first-order condition with respect to k' is:

$$c^{-\sigma} = \beta V'(k')$$

Both have the same form because they both involve choosing k' optimally. Using the envelope condition on the working problem:

$$V_k^W(k) = u'(c)[f_k(k, \bar{h}) + (1-\delta)] = c^{-\sigma}\left[\alpha k^{\alpha-1}\bar{h}^{1-\alpha} + (1-\delta)\right]$$

Using the envelope condition on the not-working problem:

$$V_k^N(k) = u'(c)(1-\delta) = c^{-\sigma}(1-\delta)$$

$V'(k_{t+1})$ depends on what the household will **optimally choose tomorrow**. Since $V(k) = \max\{V^W(k), V^N(k)\}$, the derivative is:

$$V'(k_{t+1}) = \begin{cases} V_k^W(k_{t+1}) & \text{if } V^W(k_{t+1}) > V^N(k_{t+1}) \text{ (work tomorrow)} \\ V_k^N(k_{t+1}) & \text{if } V^N(k_{t+1}) > V^W(k_{t+1}) \text{ (don't work tomorrow)} \end{cases}$$

Substituting the envelope conditions:

$$V'(k_{t+1}) = \begin{cases} c_{t+1}^{-\sigma}\left[\alpha k_{t+1}^{\alpha-1}\bar{h}^{1-\alpha} + (1-\delta)\right] & \text{if work tomorrow} \\ c_{t+1}^{-\sigma}(1-\delta) & \text{if don't work tomorrow} \end{cases}$$

Combining to get the Euler equation:

$$c_t^{-\sigma} = \beta \times \begin{cases} c_{t+1}^{-\sigma}\left[\alpha k_{t+1}^{\alpha-1}\bar{h}^{1-\alpha} + (1-\delta)\right] & \text{if } h_{t+1}^* = \bar{h} \\ c_{t+1}^{-\sigma}(1-\delta) & \text{if } h_{t+1}^* = 0 \end{cases}$$

where h_{t+1}^* is the optimal labor choice tomorrow.

Notice several important features:

- **Today's labor choice** affects c_t through the budget constraint, but the Euler equation form is the same regardless of today's choice. This is because labor and consumption are additively separable.
- **Tomorrow's labor choice** affects the return on saving:
  - If working tomorrow: capital earns marginal product $\alpha k_{t+1}^{\alpha-1}\bar{h}^{1-\alpha}$ plus undepreciated value $(1-\delta)$
  - If not working tomorrow: capital only retains undepreciated value $(1-\delta)$
- The household saves more today if they expect to work tomorrow (higher return on capital)

(d) Derive the condition under which the household chooses to work ($h = \bar{h}$) versus not work ($h = 0$). This should be an inequality involving $V^W(k)$ and $V^N(k)$.

Provide economic intuition: what factors make working more attractive? How does the capital stock affect this decision?

ANSWER: The household chooses to work if and only if:

$$V^W(k) \geq V^N(k)$$

This defines a threshold capital level $\bar{k}$ such that:

- If $k < \bar{k}$: household works ($h^* = \bar{h}$)
- If $k \geq \bar{k}$: household does not work ($h^* = 0$)

Plugging in the value function and letting c^W and c^N represent consumption when working and not working

$$\frac{(c^W)^{1-\sigma}}{1-\sigma} - \chi\bar{h} + \beta V(k') \geq \frac{(c^N)^{1-\sigma}}{1-\sigma} + \beta V(k')$$

$$\frac{(c^W)^{1-\sigma}}{1-\sigma} - \chi\bar{h} \geq \frac{(c^N)^{1-\sigma}}{1-\sigma}$$

When k is low consumption is low and the marginal utility of consumption is high. When the agent works the gain in potential consumption is $k^\alpha(\bar{h})^{(1-\alpha)}$ and for low k the marginal utility of consumption outweighs the cost of working $\chi\bar{h}$. As k increases, consumption in both states increases, and the marginal utility of consumption decreases. Eventually the marginal utility of consumption falls below the cost of working. Therefore, there exists some cutoff level of capital $\bar{k}$ above which the agent does not work.

To make working more attractive the cost of working must decline, either through a lower χ or a lower $\bar{h}$.

(e) Consider the steady state where $k_{t+1} = k_t = k^*$ and $c_{t+1} = c_t = c^*$.

(i) Suppose the household works in steady state ($h^* = \bar{h}$). Derive the steady state capital-output ratio $k^*/f(k^*, h^*)$ as a function of the model parameters. How does it compare to the baseline Ramsey model without labor choice (you solve for this in PS3)?

ANSWER: In steady state with $h^* = \bar{h}$, we have $k' = k = k^*$ and $c' = c = c^*$.
From the Euler equation and envelope condition:

$$1 = \beta\left[\alpha(k^*)^{\alpha-1}\bar{h}^{1-\alpha} + (1-\delta)\right]$$

Solving for k^*:

$$\alpha(k^*)^{\alpha-1}\bar{h}^{1-\alpha} = \frac{1}{\beta} - (1-\delta) = \rho + \delta$$

where $\rho = \frac{1-\beta}{\beta}$.

$$k^* = \left[\frac{\alpha \bar{h}^{1-\alpha}}{\rho + \delta}\right]^{\frac{1}{1-\alpha}}$$

The capital-output ratio is:

$$\frac{k^*}{f(k^*, h^*)} = \frac{k^*}{(k^*)^\alpha \bar{h}^{1-\alpha}} = \frac{1}{(k^*)^{\alpha-1}\bar{h}^{1-\alpha}} = \frac{\alpha}{\rho + \delta}$$

This is identical to the baseline Ramsey model. The discrete labor choice affects the level of capital, but the capital-output ratio in steady state is unchanged.

(ii) Now suppose the household does not work in steady state ($h^* = 0$). What is the steady state capital stock? Is this a stable steady state?

ANSWER: If $h^* = 0$ in steady state, then there is no production, only depreciation. The budget constraint becomes:

$$c^* = (1-\delta)k^* - k^* = -\delta k^*$$

This is impossible since consumption must be positive. Therefore, a non-working steady state with positive capital does not exist. The only non-working steady state is $k^* = 0$, $c^* = 0$, which is unstable and not economically interesting.

(iii) Under what parameter conditions would we expect a working steady state? Provide a condition involving χ, $\bar{h}$, and other parameters.

ANSWER: For a working steady state to be optimal, we need:

$$V^W(k^*) \geq V^N(k^*)$$

This requires that the benefit of working (additional output) exceeds the cost:

$$\frac{(c_W^*)^{1-\sigma}}{1-\sigma} - \chi\bar{h} + \beta V(k^*) \geq \frac{(c_N^*)^{1-\sigma}}{1-\sigma} + \beta V(k^*)$$

where $c_W^* = (k^*)^\alpha \bar{h}^{1-\alpha} + (1-\delta)k^* - k^*$ and $c_N^* = (1-\delta)k^* - k^*$.
In steady state, this simplifies to ($c_N^* = 0$ since consumption can't be negative):

$$\frac{(c_W^*)^{1-\sigma}}{1-\sigma} \geq \chi\bar{h}$$

A working steady state is more likely when:

- χ is low (working is not too costly)
- $\bar{h}$ is low (don't have to work too many hours)
- α is low (labor is very productive)

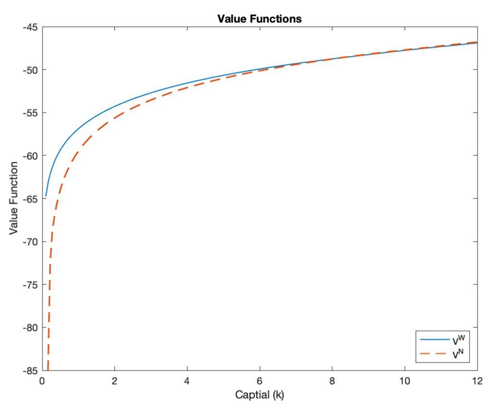
2 Computational Implementation

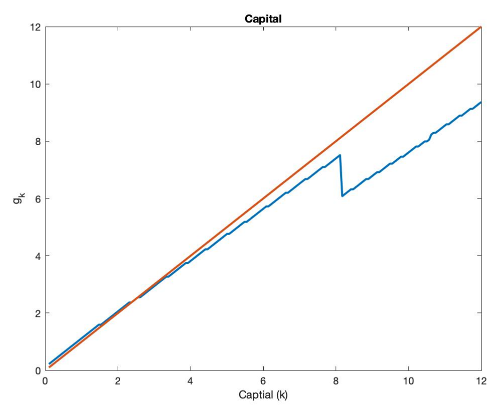
(a) What is the state space and control space? What do we need an initial guess for? In what order should you solve the two value functions $V^W(k)$, $V^N(k)$, and $V(k)$?

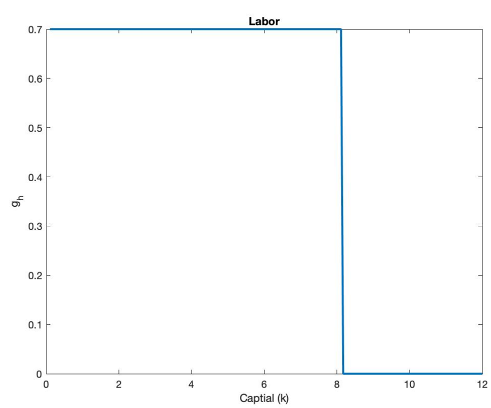
ANSWER: The state space is capital $k \in [k_{\min}, k_{\max}]$, discretized into a grid of N points. Control space is next period capital $k' \in [k_{\min}, k_{\max}]$ and labor $h \in \{0, \bar{h}\}$. We only need an initial guess for $V(k)$, since $V^W(k)$ and $V^N(k)$ depend on the full value function and not on themselves. We will first solve for $V^W(k)$ and $V^N(k)$ and then we can solve for $V(k)$.

(b) Write a MATLAB script to solve the model using value function iteration. The script should produce, the converged value functions and the policy functions. Plot $V^W(k)$ and $V^N(k)$ on the same graph. Plot the optimal labor choice $g_h(k)$ as a function of k. Plot the capital policy function $g_k(k)$ along with the 45-degree line to identify steady states.

ANSWER:







(c) At what value of k does the worker become indifferent between working and not working ($\bar{k}$)? Does what you find make sense? What is the value of the steady state that comes out of the VFI and how does it compare to the analytical value?

ANSWER: $\bar{k} = 8.17$. Below this value agents choose to work and above this value agents do not work. This makes sence because rich household (above $\bar{k}$) prefer not to work because they can afford to pay the cost of zero labor income by consuming their wealth.

The analytical value of steady state capital is $k^* = 2.473$, in our code we have three values on the k-grid where capital does not change, this comes from out grid size and lack of interpolation. If we average the values where capital does not change we get $g_k(k) = k = 2.4621$.