EC9A2 Problem Set 5

Model Setup

Consider an economy populated by a representative household that maximizes expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where the period utility function is:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

Output is produced using capital and labor according to:

$$y_t = A_t k_t^{\alpha}$$

Labor is normalized to 1 and suppressed from the notation. Capital evolves according to:

$$k_{t+1} = (1-\delta)k_t + i_t$$

where $\delta \in (0,1)$ is the depreciation rate and i_t is investment. The resource constraint is:

$$c_t + i_t = y_t$$

TFP follows a two-state Markov chain. Let $A_t \in \{A_L, A_H\}$ where $A_L < A_H$ represent low and high productivity states. The transition probability matrix is:

$$\Pi = \begin{pmatrix} \pi_{LL} & \pi_{LH} \\ \pi_{HL} & \pi_{HH} \end{pmatrix}$$

where $\pi_{ij} = \Pr(A_{t+1} = A_j | A_t = A_i)$ and $\pi_{iL} + \pi_{iH} = 1$ for $i \in \{L, H\}$.

Parameters:

- Time period: one quarter
- $\beta = 0.99$ (discount factor)
- $\sigma = 2$ (risk aversion)
- $\alpha = 0.33$ (capital share)
- $\delta = 0.025$ (depreciation rate)
- $A_L = 0.95$, $A_H = 1.05$ (TFP states)
- Transition probabilities: $\pi_{LL} = 0.9$, $\pi_{HH} = 0.9$

1 Theoretical Analysis

(a) Write down the Bellman equation for this economy. Be explicit about the state variables.

(b) Derive the Euler equation for this problem. Interpret the economic meaning of each term.

(c) How does uncertainty affect the intertemporal consumption-saving decision compared to the deterministic model?

(d) Rewrite the problem using two Bellman equations, one for high productivity value V^H and own for the low productivity value V^L.

(e) What is the average duration of a recession (low productivity state) implied by the parameter values?

(f) Calculate the stationary distribution of the TFP process. What fraction of time does the economy spend in each state in the long run? What is the unconditional mean of TFP?

2 Numerical Solution

(a) Consider the problem you wrote down in Question 1d, what is the state space and control space? What do we need an initial guess for? How many value functions and policy functions does this problem have?

(b) Write a MATLAB script to solve the model using value function iteration. The script should produce, the converged value functions and the policy functions. Plot V^H and V^L on the same graph. Plot the capital policy functions $g_k^H(k)$ and $g_k^L(k)$ along with the 45-degree line to identify steady states. Compute and report the target capital stock for each TFP state (where $g_i(k) = k$ for $i \in \{H, L\}$).

HINT: For the captial grid, calculate what the value of steady state capital would be in a deterministic version of the model using the parameter values and the mean TFP you solved for in Question 1f. Then create a grid $[0.5k_{ss}^{det}, 1.5k_{ss}^{det}]$.

(c) Simulate the economy for $T = 200$ periods starting from the low-productivity steady state. Generate a random sequence of TFP shocks consistent with the Markov transition matrix. (Use $rng(123)$ to set the seed.) Plot the time series for output, consumption, investment, and capital over the simulation period.

(d) Compute and report the following statistics:

(i) Mean and standard deviation of output, consumption, and investment

(ii) Correlation between output and consumption

(iii) Correlation between output and investment

(iv) Autocorrelation of output