EC9A2 Problem Set 5

Model Setup

Consider an economy populated by a representative household that maximizes expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where the period utility function is:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

Output is produced using capital and labor according to:

$$y_t = A_t k_t^{\alpha}$$

Labor is normalized to 1 and suppressed from the notation. Capital evolves according to:

$$k_{t+1} = (1-\delta)k_t + i_t$$

where $\delta \in (0,1)$ is the depreciation rate and i_t is investment. The resource constraint is:

$$c_t + i_t = y_t$$

TFP follows a two-state Markov chain. Let $A_t \in \{A_L, A_H\}$ where $A_L < A_H$ represent low and high productivity states. The transition probability matrix is:

$$\Pi = \begin{pmatrix} \pi_{LL} & \pi_{LH} \\ \pi_{HL} & \pi_{HH} \end{pmatrix}$$

where $\pi_{ij} = \Pr(A_{t+1} = A_j | A_t = A_i)$ and $\pi_{iL} + \pi_{iH} = 1$ for $i \in \{L, H\}$.

Parameters:

- Time period: one quarter
- $\beta = 0.99$ (discount factor)
- $\sigma = 2$ (risk aversion)
- $\alpha = 0.33$ (capital share)
- $\delta = 0.025$ (depreciation rate)
- $A_L = 0.95$, $A_H = 1.05$ (TFP states)
- Transition probabilities: $\pi_{LL} = 0.9$, $\pi_{HH} = 0.9$

1 Theoretical Analysis

(a) Write down the Bellman equation for this economy. Be explicit about the state variables.

ANSWER: The state variables are the current capital stock k and the current TFP state $A \in \{A_L, A_H\}$. The Bellman equation is:

$$V(k, A) = \max_{k'} \{u(c) + \beta \mathbb{E}[V(k', A')|A]\}$$

subject to

$$c = Ak^\alpha + (1 - \delta)k - k'$$

where the expectation is taken over the next period's TFP state A' conditional on the current state A. More explicitly:

$$V(k, A) = \max_{k' \geq 0} \left\{ \frac{(Ak^\alpha + (1 - \delta)k - k')^{1-\sigma}}{1 - \sigma} + \beta \sum_{A' \in \{A_L, A_H\}} \pi(A'|A)V(k', A') \right\}$$

(b) Derive the Euler equation for this problem. Interpret the economic meaning of each term.

ANSWER: Taking the first-order condition with respect to k':

$$u'(c_t) = \beta \mathbb{E}_t[V_k(k_{t+1}, A_{t+1})]$$

where V_k denotes the partial derivative of the value function with respect to capital. To find $V_k(k, A)$, we differentiate the Bellman equation with respect to k (Envelop Condition):

$$V_k(k, A) = u'(c)[A\alpha k^{\alpha-1} + 1 - \delta]$$

Combining the FOC and envelope condition by advancing one period:

$$u'(c_t) = \beta \mathbb{E}_t[u'(c_{t+1})(A_{t+1}\alpha k_{t+1}^{\alpha-1} + 1 - \delta)]$$

With CRRA preferences:

$$c_t^{-\sigma} = \beta \mathbb{E}_t[c_{t+1}^{-\sigma}(A_{t+1}\alpha k_{t+1}^{\alpha-1} + 1 - \delta)]$$

Economic Interpretation:

- The left-hand side represents the marginal utility cost of saving one additional unit today (foregone consumption).
- The right-hand side represents the expected discounted marginal utility benefit of that saving tomorrow.

- The term $A_{t+1}\alpha k_{t+1}^{\alpha-1} + 1 - \delta$ is the gross return on capital: the marginal product of capital plus the undepreciated portion.
- At the optimum, the household is indifferent between consuming today and saving for tomorrow, accounting for the uncertain return on capital.

(c) How does uncertainty affect the intertemporal consumption-saving decision compared to the deterministic model?

ANSWER: Uncertainty affects the consumption-saving decision through several channels:

(a) **Precautionary Saving**: With risk aversion ($\sigma > 0$) and uncertain future productivity, households have an incentive to save more as a buffer against bad states. This is due to the convexity of the marginal utility function.

(b) **Expected Returns**: The household must form expectations about future returns to capital, which depend on the uncertain TFP realization. The expectation operator in the Euler equation captures this.

(c) **State-Dependent Policy**: Unlike the deterministic model with a unique policy function, here the optimal saving decision depends on the current productivity state. The household saves differently in high vs. low productivity states.

(d) **No Closed-Form Solution**: The presence of uncertainty generally eliminates the possibility of analytical solutions, requiring numerical methods.

(d) Rewrite the problem using two Bellman equations, one for high productivity value V^H and own for the low productivity value V^L.

ANSWER: We can write separate value functions for each productivity state:

$$V^H(k) = \max_{k'} \left\{ \frac{(A_H k^\alpha + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} + \beta[\pi_{HH}V^H(k') + \pi_{HL}V^L(k')] \right\}$$

$$V^L(k) = \max_{k'} \left\{ \frac{(A_L k^\alpha + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} + \beta[\pi_{LH}V^H(k') + \pi_{LL}V^L(k')] \right\}$$

(e) What is the average duration of a recession (low productivity state) implied by the parameter values?

ANSWER: If in a recession, the probability that we exit the recession in n periods is:

$$P(duration = n) = \pi_{LL}^{n-1}(1-\pi_{LL})$$

The expecyed duration of staying in the low productivity state is given by:

$$
\begin{aligned}
E[Duration] &= (1-\pi_{LL})\sum_{n=1}^{\infty} n\pi_{LL}^{n-1} \\
&= (1-\pi_{LL})\frac{1}{(1-\pi_{LL})^2} \\
&= \frac{1}{1-\pi_{LL}}
\end{aligned}
$$

Plugging in the parameter values:

$$
\text{Duration} = \frac{1}{1-\pi_{LL}} = \frac{1}{1-0.9} = \frac{1}{0.1} = 10 \text{ quarters}
$$

Therefore, the average recession lasts 10 quarters (2.5 years).

(f) Calculate the stationary distribution of the TFP process. What fraction of time does the economy spend in each state in the long run? What is the unconditional mean of TFP?

ANSWER: Let $\mu = (\mu_L, \mu_H)$ denote the stationary distribution. It must satisfy:

$$
\mu = \mu\Pi
$$

and $\mu_L + \mu_H = 1$. This gives us:

$$
\begin{aligned}
\mu_L &= 0.9\mu_L + 0.1\mu_H \\
\mu_H &= 0.1\mu_L + 0.9\mu_H
\end{aligned}
$$

From the first equation: $0.1\mu_L = 0.1\mu_H$, so $\mu_L = \mu_H$. Combined with $\mu_L + \mu_H = 1$, we get:

$$
\mu_L = \mu_H = 0.5
$$

The economy spends 50% of time in each state in the long run.

The unconditional mean of TFP is:

$$
\mathbb{E}[A] = \mu_L \cdot A_L + \mu_H \cdot A_H = 0.5(0.95) + 0.5(1.05) = 1.0
$$

2 Numerical Solution

(a) Consider the problem you wrote down in Question 1d, what is the state space and control space? What do we need an initial guess for? How many value functions and policy functions does this problem have?

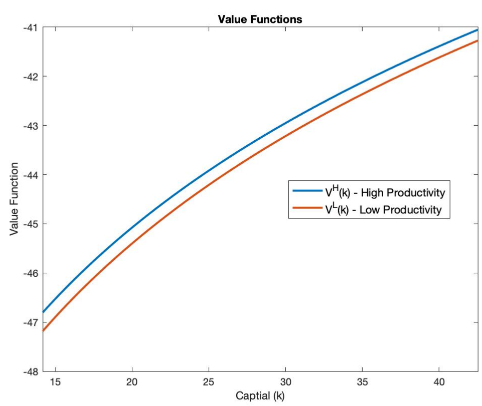
ANSWER:

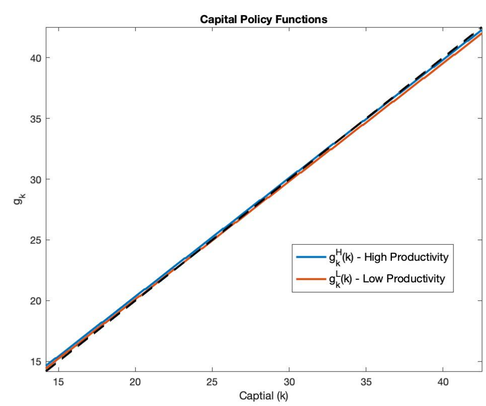
- **State Space**: Capital stock $k \in [k_{\min}, k_{\max}]$ (continuous, discretized into a grid)
- **Control Space**: Next period's capital $k' \in [0, Ak^{\alpha} + (1-\delta)k]$ (bounded by the resource constraint)
- **Initial Guess**: We need an initial guess for both value functions $V^H(k)$ and $V^L(k)$
- **Number of Functions**:
  - 2 value functions: $V^H(k)$ and $V^L(k)$
  - 2 policy functions: $g_k^H(k)$ and $g_k^L(k)$

(b) Write a MATLAB script to solve the model using value function iteration. The script should produce, the converged value functions and the policy functions. Plot V^H and V^L on the same graph. Plot the capital policy functions $g_k^H(k)$ and $g_k^L(k)$ along with the 45-degree line to identify steady states. Compute and report the target capital stock for each TFP state (where $g_k^i(k) = k$ for $i \in \{H, L\}$).

HINT: For the captial grid, calculate what the value of steady state capital would be in a deterministic version of the model using the parameter values and the mean TFP you solved for in Question 1f. Then create a grid $[0.5k_{ss}^{det}, 1.5k_{ss}^{det}]$.

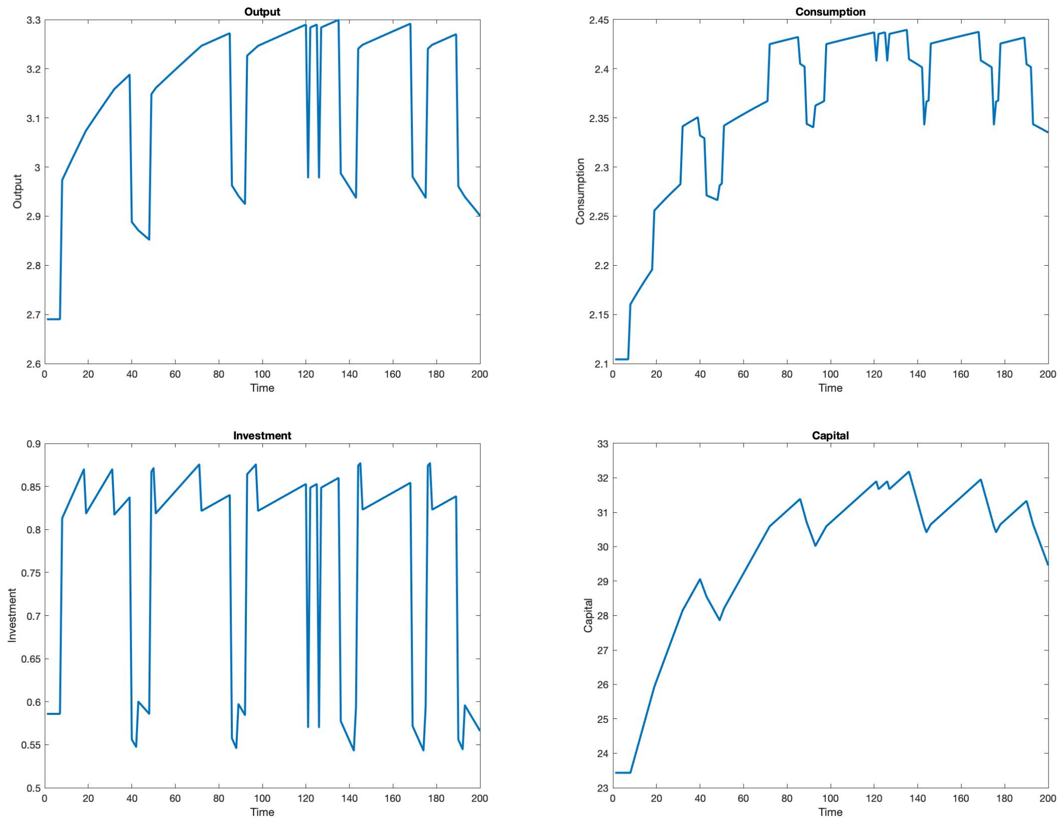
ANSWER: The target level of capital is 33.8590 for the high state and 23.4343 for the low state.





(c) Simulate the economy for $T = 200$ periods starting from the low-productivity steady state. Generate a random sequence of TFP shocks consistent with the Markov transition matrix. (Use $rng(123)$ to set the seed.) Plot the time series for output, consumption, investment, and capital over the simulation period.

ANSWER:



(d) Compute and report the following statistics:

(i) Mean and standard deviation of output, consumption, and investment

(ii) Correlation between output and consumption

(iii) Correlation between output and investment

(iv) Autocorrelation of output

ANSWER:

Variable	Mean	Std. Deviation
Output	3.1376	0.1617
Consumption	2.3641	0.0862
Investment	0.7735	0.1187
Correlations		
$\rho(Y, C)$		0.6995
$\rho(Y, I)$		0.8547
Autocorrelations		
$\rho(Y_t, Y_{t-1})$		0.8716

Table 1: Business Cycle Statistics